SUSY Revision Class (15/4)

Review: Topics

Supersymmetry

- Coleman-Madula Theorem
- SUSY Algebra.

$$\{Q, Q\} \sim P.$$

Spinors + Fermions

$SL(2,\mathbb{C}) \longrightarrow SO(1,3)$.

- Fundamental reps of $SL(2,\mathbb{C})$.

Reps.

Poincaré ~ $\mathcal{P}$ ⇒ SuperPoincaré

$$\hat{\mathcal{P}} = \mathcal{P} \times \text{"}\underbrace{\text{ferm}}\text{"}$$

"Q" → raising/lowering operators.

Starting point.

$|\Omega_0\rangle$

Clifford vacuum

1) Massless

- 1 set generators ⇒ "half-multiplets"

2) Massive

- 2 sets Q_1, Q_2 ⇒ full massive multiplets
- $Z \neq 0 \Rightarrow M_i - 2Z_i \geq 0$ ← BPS multiplets

Long → Short → Ultrashort.

Theories + Actions

SUSY Multiplet → Action

$$(\phi, \psi, \underline{F})$$

↑ aux field.

1) Component fields.

$$\phi : M^{1,3} \longrightarrow X .$$

2) Superfields.

$$\Phi : \hat{M}^{1,3} \longrightarrow X$$

$\hat{M}^{1,3}$ superspace

$M^{1,3}$ "x" "fermion"

$\theta, \bar{\theta}$

4d $N=1$ SUSY.

Superfields is the way to go.

Actions

$$S = \int d^4x \, \mathcal{L}$$

D-term

collection of superfields.

$$\mathcal{L}_D = \int d^2\theta d^2\bar{\theta} \, Y .$$

F-term

$$\mathcal{L}_F = \int d^2\theta \, Y + c.c.$$

Superfield constraints

(i) Chiral ⇒ Chiral multiplet

$$\bar{D}_{\dot{\alpha}} \Phi = 0 ,$$

Chiral theories

(ii). Real ⇒ Vector multiplet.

$$V = \bar{V}$$

Gauge theories.

Chiral Theories

Kähler potential

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\, \boxed{K(\Phi, \bar\Phi)}$$

$$+ \left(\int d^2\theta\, W(\Phi) + \text{c.c.} \right)$$

Superpotential

$$\Phi = \boxed{\phi} + \theta\psi + \ldots$$

$$[\phi] = 1$$

$$[\psi] = \frac{3}{2} \Rightarrow [\theta] = -\frac{1}{2}$$

$$Y = y + \ldots + \theta^2\bar\theta^2 D.$$

$$[Y] = d \longrightarrow [Y_D] = d + \underline{2}.$$

$\int d^2\theta\, d^2\bar\theta$ takes out $\theta^2\bar\theta^2$.

$$[K_D] = 4 \quad (\text{so } [\mathcal{L}] = 4)$$

$$\Rightarrow \boxed{[K] = 2.}$$

Similarly $\boxed{[W] = 3.}$

Renormalisability

$$\left.\begin{aligned}[K]&=2\\ [W]&=3.\end{aligned}\right\} \longrightarrow$$

renormalisable.

$W(\Phi)=\sum a_n\Phi^n$

must have $n\leq 3$.

(∵ $[a_n]<0$. ⇒ non-renorm.)

$$[K]=2 \to K=\Phi^\dagger\Phi+\dots$$

$$\mathcal{L} = \underbrace{K^{i\bar{j}}\,\partial_i\phi\,\partial_{\bar{j}}\bar{\phi}}+\dots$$

$$+\underbrace{R_{i\bar{j}k\bar{l}}\,\varphi^i\varphi^j\bar{\varphi}^{\bar{k}}\bar{\varphi}^{\bar{k}}}_{\text{non-renormalisable}}$$

Target space is Kähler

$=0$ if $R=0$ ie.

if X is $\boxed{\text{Riemann-flat}}$

Vector Superfields → Gauge theories

$$V \longmapsto V+\Omega+\bar{\Omega}$$

$$V+i(\Omega-\bar{\Omega})$$

← chiral fields.

Moduli Space → distributions of lowest-energy states.

→ scalar potentials.

$$D = \phi T^a \phi D^a + \boxed{\xi}$$

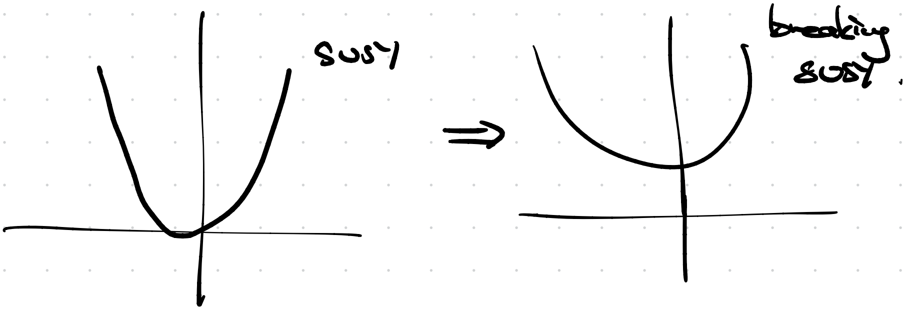
Gauge + Chiral:

$$V = \underbrace{\boxed{\bar{F}F}}_{F\text{-flat}} + \underbrace{\boxed{\frac{1}{2}D^2}}_{D\text{-flat}}$$

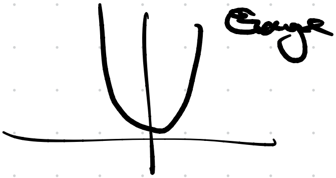
- $\langle \phi | \{Q, \bar{Q}\} | \phi \rangle \geq 0 \Rightarrow \langle \overset{\sim P^0}{H} \rangle \geq 0$.

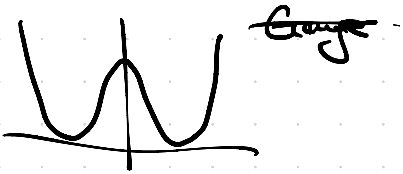
Lowest energy state $E=0$ in SUSY.

$\underline{E>0}$ ⇒ breaks SUSY.



$\langle \phi \rangle = 0$. ⟶ $\langle \phi \rangle \neq 0$.





2018

Q1

$$\begin{cases} P_\mu = -i\partial_\mu \\ Q_\alpha = -i\partial_\alpha - (\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar Q_{\dot\alpha} = i\bar\partial_{\dot\alpha} + \theta^\gamma(\sigma^\mu)_{\gamma\dot\alpha}\partial_\mu \end{cases}$$

(a).

$$[P_\mu, P_\nu] = 0.$$

$$[P_\mu, Q_\alpha] = 0.$$

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}P_\mu.$$

$\therefore$ $\mathcal{N}=1$ SUSY alg is satisfied.

(b). General superfield.

$$Y = C(x) + \underline{i\theta\chi + \overset{-i\bar\theta\bar\psi}{c.c.}} + \underline{\tfrac{i}{2}\theta\theta M + c.c.}$$

$$- \theta\sigma^\mu\bar\theta\, v_\mu(x) + \underline{i\theta\theta\bar\theta\bar\eta + c.c.}$$

$$+ \tfrac{1}{2}\theta\theta\bar\theta\bar\theta D(x).$$

Chiral:

$$\bar D_{\dot\alpha}\Phi = 0.$$

($\simeq -i\bar\partial_{\dot\alpha}$

$\bar D_{\dot\alpha}$ covariant deriv.

LH-invariant generators of superspace)

Easiest to use chiral coor. $y^\mu = \underline{x^\mu + i\theta\sigma^\mu\bar\theta}$

$$\Phi(y,\theta) = \phi(y) + \underline{\sqrt{2}}\theta\psi(y) + \theta\theta F(y)$$

∴ This furnishes a chiral multiplet.

(c).

Component field	Spin	Name
ϕ	0	ex. scalar
ψ	$\frac{1}{2}$	Weyl ferm
F	0	aux. field

General action? $S = \int d^4x\, \mathcal{L}$

$$\mathcal{L} = \int d^2\theta d^2\bar\theta\, K(\Phi, \bar\Phi)$$

$$+\left(\int d^2\theta\, W(\Phi) + c.c.\right)$$

$+|F|^2 - - .$

$$= \partial_\mu\phi\overline{\partial_\mu\phi} + i\bar\psi\sigma^\mu\partial_\mu\psi + W(\phi) + \dots$$

(after aux. F-field is integrated out)

(d). Φ , $\underline{\Phi \longmapsto \Phi + i}$ axionic shift symmetry.

- Kähler potential → not affected
- Superpotential

$W(\Phi)$ × not allowed perturbatively.

powers

$$W = \sum_{n=1}^{\infty} a_n \Phi^n \quad \times \text{ not allowed.}$$

(e). To avoid this → non-perturbative superpotentially

$$W = \Lambda^3 e^{-2\pi a \Phi}.$$

$$\Phi \longmapsto \Phi + i.$$

$$W = \Lambda^3 e^{-2\pi a \Phi}$$ unchanged to leading $\theta, \bar{\theta}$ order.

$$K = \Phi^\dagger \Phi$$

$$F_\Phi = 0 = W_\Phi + K_\Phi W = 0,$$

$$(-2\pi a + \Phi^\dagger) W = 0$$

$$\Rightarrow \left(\Phi^\dagger = 2\pi a\right)$$ SUSY vacuum

Q2

(a). Coleman-Mandula Theorem.

(i) only finite # particles
(ii) energy gap bt. vacuum + 1P level
(iii) Nontrivial scattering.

→ Symmetry : Poincaré × Internal.

⇒ Bosonic generator

SUSY avoids this ⇒ fermionic generators

$$\underline{\{Q, P]} .$$ ✓

(b). $\underline{SL(2,\mathbb{C}) \to SO(3,1)}$ (Q3 Sheet 1)

Construct a map:

$$\tilde{\varphi} : SO(3,1) \to \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$$

$$\sigma_\mu = (\mathbb{1}_2, \underline{\sigma^i}) \qquad \bar{\sigma}_\mu = (\mathbb{1}_2, -\sigma^i) .$$

$$tr(\sigma_\mu \bar{\sigma}_\nu) = 2 g_{\mu\nu} .$$

2×2 matrix A (This is Sheet 1 Q3.)

$$A = \frac{1}{2}\mathrm{tr}(\bar{\sigma}^\mu A)\sigma_\mu .$$

x^μ : $\underline{\underline{x}}$ 2×2 matrix :

$$\underline{\underline{x}} = \sigma_\mu x^\mu .$$

$$\underline{\underline{x}} \longmapsto \underline{\underline{x}}' = A\underline{\underline{x}}A^\dagger .$$

$$x^\mu \longmapsto \Lambda^\mu{}_\nu x^\nu . \qquad (\sigma_\mu \Lambda^\mu{}_\nu = A\sigma_\nu A^\dagger)$$

$$\Rightarrow \Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{tr}(\bar{\sigma}^\mu A \sigma_\nu A^\dagger) .$$

$$\sigma_\nu A^\dagger \bar{\sigma}^\nu = 2\mathrm{tr}(A^\dagger)\mathbb{1} . \Rightarrow \Lambda^\mu{}_\mu = \mathrm{tr}|A|^2$$

$$\sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu = 2\mathrm{tr}(A^\dagger)A .$$

$$\Rightarrow \boxed{A = \pm \frac{\sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu}{2\sqrt{\Lambda^\mu{}_\mu}} .}$$

$SL(2,\mathbb{C})$ $\mathbb{Z}_2$ $SO(1,3)$.

(c).
$$\left.\begin{array}{l}[P,P]\\ [P,M]\end{array}\right\}\ \text{Poincaré}$$

$$[P,Q]=0$$

$$\{\underset{-}{Q},\underset{\cdot}{Q}\}\sim \underset{\uparrow\ i\sigma}{P}_{\alpha\dot\alpha}$$

R-symmetry is an $\underline{\underline{\text{outer}}}$ automorphism of SUSY alg.

$$\Big(\ \underline{Q_\alpha \xrightarrow{R} e^{-ia}Q_\alpha}\ .\Big)$$

4d $\mathcal{N}=1$

Sheet 4 Q1.

Superfield $Y \longmapsto e^{-ira}\,Y$.

$R[Y]=r$.   e.g.

$$\Rightarrow\ \begin{cases} R[Y|_0]=r\\ R[Y|_\theta]=r-1.\\ R[Y|_{\theta\theta}]=r-2.\end{cases}\qquad \begin{array}{l} R[\phi]=r\\ R[\psi]=r-1\\ R[F]=r-2\end{array}$$

[(d). Discrete R-symmetry. $Q\mapsto -Q$.

R-parity   → forbids R-parity violating terms.

→ stable light particles.]

(e). $\mathcal{N}=4$ SUSY in 4d

$$\to \quad \underline{U(\mathcal{N})}. \Rightarrow \underline{\underline{U(4)}},$$ ✓

Q3. SUSY Abelian gauge → vector superfield.

(i). Vector superfield.

. real: $V = \bar{V}$,

• $V = C + i\theta\varphi + (c.c.) + \dots$

(ii) Gauge Transformation

$$\cdot\ V \longmapsto V + (\Phi + \bar{\Phi})$$

Φ chiral field.

cpt. fields transf.

$$C \longmapsto C + 2\mathrm{Re}\phi \quad \Rightarrow \quad \mathrm{Re}\phi = -\frac{c}{2}$$

$$\chi \longmapsto \chi - i\sqrt{2}\psi.$$

$$M \longmapsto M - 2\mathrm{Im}(F)$$

$$N \longmapsto N + 2\mathrm{Re}F$$

$$D \longmapsto D$$

$$\lambda \longmapsto \lambda$$

$$\boxed{v^\mu \longmapsto v^\mu - 2\partial^\mu \mathrm{Im}\phi}$$

Gauge fields → easiest to fix WZ gauge

→ to get rid of C, χ, M, N

$\circ \quad \theta \quad \theta\theta \quad \bar\theta\bar\theta$

$$V_{WZ} = \theta\sigma^\mu\bar\theta \underbrace{v_\mu}_{\text{gauge}} + i\theta\theta\bar\theta\underbrace{\bar\lambda(x)}_{\text{gaugino}} - i\bar\theta\bar\theta\theta\lambda(x)$$

$$+ \frac{1}{2}\theta\theta\bar\theta\bar\theta \underbrace{D(x)}_{\text{aux. field}}.$$

Field strength superfield

$$W_\alpha = \bar D^2 D_\alpha V.$$

$$\Rightarrow \int d^2\theta\, W^\alpha W_\alpha \quad (D\text{-term}).$$

$$= -\frac{1}{2}\underbrace{F^2} - \underbrace{2i\lambda\sigma^\mu\partial_\mu\bar\lambda} + D^2 + (F\wedge F)$$

Gauge-inv.

$$V \mapsto (\Omega + \bar\Omega) + V. \quad \text{because} \begin{cases} \bar D_{\dot\alpha}\Omega = 0 \\ D_\alpha\bar\Omega = 0. \end{cases}$$

$$W_\alpha \mapsto W_\alpha$$

Form of the action.

Superfield:

$$\int d^2\theta\, W^\alpha W_\alpha + c.c.$$

cpt:

$$\int d^2\theta\, W^\alpha W_\alpha + c.c.$$

$$= -F^2 - 4i\lambda\sigma^\mu\partial_\mu\bar\lambda + 2D^2.$$

$$U(1)\ \overset{V}{\text{gauge}} \longleftrightarrow \boxed{\Phi}.$$

$$\underline{W = m\Phi^2}$$

(b). Superpotential?

$$\Phi \longmapsto \boxed{e^{-i\alpha}\Phi}.$$

spurion

$$\Phi_1 \longmapsto e^{-i\alpha}\Phi_1$$

$$\Phi_2 \longmapsto e^{i\alpha}\Phi_2$$

~~Either the couplings transform under gauge~~

Or $\boxed{W = 0}$.

$$\underline{W = m\Phi_1\Phi_2}$$

(c). F.I. term

$$W = \sum a_{ijk\ldots}\,\Phi_1^i\Phi_2^j\Phi_3^k$$

R_1 R_2 R_3

$$\mathcal{L} = \frac{1}{2}\xi D = \frac{1}{2}\xi\int d^2\theta\, d^2\bar\theta\, V.$$

$$1 \subset \underbrace{R_1 \times \ldots \times R_1}\ \ldots$$

$$V \longmapsto V + \Omega + \bar\Omega$$

$$D \longmapsto D + \partial^2(\quad)$$

(d).

(Non-abelian X.

$$[\mathrm{tr}\, D] \longrightarrow [\mathrm{tr}\, D^a T^a] = 0.$$

for semisimple Lie alg.)

$$D^a = | \bar{\Phi}_i \underline{(T^a)^i}_j \phi^j + \xi |$$

$$V \supset D^2 = (g|\phi|^2 + \xi)^2$$

$g>0,\ \xi>0 \Rightarrow V>0 \Rightarrow$ SUSY-broken vacua

$g>0,\ \xi<0 \Rightarrow |\phi|^2 \sim \frac{\xi}{g}$ SUSY-preserving Gauge-broken vacua.

D-term breaking model.

2022 Q1

SUSY QM

$$\begin{cases} Q = i\left(\frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial t}\right) \\ \bar{Q} = i\left(\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}\right) \\ H = i\frac{\partial}{\partial t}. \end{cases}$$

$$\underline{\{Q, \bar{Q}\} = 2H}$$

$$LH = \underbrace{i\frac{\partial}{\partial t}}_{Q\bar{Q}} + \underbrace{i\frac{\partial}{\partial t}}_{\bar{Q}Q} = 2H = RHS.$$

$$X = x(t) + \underline{\theta\psi(t)} \ \underline{- \bar{\theta}\bar{\psi}(t)} \ \underline{+ \theta\bar{\theta}D(t)}$$

(i).

$$L_{kin} = \int d\theta d\bar{\theta} \left(-\frac{1}{2}\bar{D}X DX\right)$$

$$D = \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}.$$

$$\Rightarrow DX = \psi + \bar{\theta}D + i\bar{\theta}\dot{x} + i\bar{\theta}\theta\dot{\psi}$$

$$\bar{D}X = -\bar{\psi} - \theta D + i\theta\dot{x} - i\theta\bar{\theta}\dot{\psi}$$

$$\Rightarrow \mathcal{L}_{kin} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}D^2 + \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi).$$

(i) $h(X)$ superpotential of SQM.

$$h'(X) = h'(x) + h''(x)(\theta\psi - \bar\theta\bar\psi)$$

$$h(X)\big|_{\theta\bar\theta} = \theta h'(X)[\psi + \bar\theta D]$$

$$\boxed{-\bar\theta\bar\psi}$$

$$= \theta\bar\theta\left[-h''(x)\bar\psi\psi + h'(x)D\right]$$

$$\therefore -\int d\bar\theta d\theta\, h(x) = h''(x)\bar\psi\psi - h'(x)D$$

(ii)

$$\mathcal{L}_{tot} = \int d\bar\theta d\theta\left(-\frac{1}{2}\bar{D}X\,DX - h(X)\right)$$

$$h'(x) = \frac{1}{2}D.$$

$$= \frac{1}{2}\dot{x}^2 + \underbrace{\frac{1}{2}D^2}_{} + \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - \underbrace{h'(x)D}_{}$$
$$+ h''(x)\bar\psi\psi.$$

$$= \underbrace{\frac{1}{2}\dot{x}^2}_{\text{KE}} - \underbrace{\frac{1}{2}(h'(x))^2}_{\text{potential scalar}} + \underbrace{\frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi)}_{} \text{KE}$$
$$+ \underbrace{h''(x)\bar\psi\psi}_{\text{interaction}}.$$

(David Skinner's §3).

Non-renormalisation Theorem

$$W = W(m, \underset{\underset{\sim\Phi^2}{\uparrow}}{\,}\!\!\!\!\! \lambda, \Phi)$$

$$m \uparrow \sim\Phi^2 \qquad \lambda \uparrow \sim\Phi^3$$

	$U(1)_B$	$U(1)_F$
Φ	1	1
m	0	-2
λ	-1	-3 .

$$W_{\text{tree}} = \frac{1}{2} m\Phi^2 + \frac{1}{3}\lambda\Phi^3 , \dots \quad \Phi^n$$

$$R[W] = 2$$

$$\Rightarrow \boxed{W = m\Phi^2 \underbrace{f\left(\frac{\lambda\Phi}{m}\right)}_{x}}$$

$$1 + x + x^2 + \dots$$

$m \to 0$:

$$W = m\Phi^2 + \lambda\Phi^3$$

[§3.3.2
Tong's
p.73-75]