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# Supersymmetry Example Classes 2026

**Lucas T. Y. Leung**[a]

[a] *Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Parks Road, Oxford OX1 3PU, UK*

*E-mail:* lucas.leung@physics.ox.ac.uk

Abstract: A set of class notes for the Supersymmetry and Supergravity course at Oxford in Hilary Term 2026.

## Contents

## 0 Introduction

These set of notes are for the Supersymmetry and Supergravity classes in 2026.

Supersymmetry, in my opinion, is still a very important topic to study. It is still used in many BSM phenomenology research, mathematical research and of course, string theory. Below is my very poor attempt to complement Michèle's amazing course which is based mostly on [1]. I wanted to give a more complete picture in the classes, so below is my rudimentary attempt to do that.

This set of notes is riddled with mistakes, so please let me know if you find any. A quick health warning too — most of the appendices are really just my own notes (because I like maths and physics) so if you don't like formal stuff, don't read anything beyond Appendix A. I don't want to be responsible for anyone getting hurt.

## 1 Spurions, Naturalness, the Hierarchy Problem and SUSY

Supersymmetry is the symmetry between fermions and bosons. To put it simply, there is a natural splitting of the Hilbert space $\mathcal{H}$ as,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F \ , \tag{1.1}$$

where $\mathcal{H}_B$ and $\mathcal{H}_F$ indicates the Hilbert space with an even and odd number of fermionic excitations respectively. The operator $\mathcal{Q}$,

$$\mathcal{Q} : \mathcal{H}_{B,F} \to \mathcal{H}_{F,B} \tag{1.2}$$

with the following two properties:

$$\mathcal{Q}^2 = 0 \ , \tag{1.3}$$

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2H \ . \tag{1.4}$$

Here $H$ is the Hamiltonian of the theory. There are immediately two consequences of having this symmetry generated by $\mathcal{Q}$.

1. $[H, \mathcal{Q}] = 0$. This $\mathcal{Q}$ actually commutes with the Hamiltonian so it is a symmetry.
2. $\langle \Psi | H | \Psi \rangle \geq 0$ for any $|\psi\rangle \in \mathcal{H}$, which is an equality if and only if $\mathcal{Q}|\psi\rangle = 0 = \mathcal{Q}^\dagger |\psi\rangle$. This means that for a supersymmetric vacua $E_0 = 0$ [1].

Before we continue, I want to begin by asking the question — why should we study supersymmetry? To do this, let me first introduce the concept of spurions.

[1] This point, as we will see, will become important in SUSY-breaking.

## 1.1 Spurions and technical naturalness

Let us define spurions as follows.

**Definition 1.1.** A **spurion field** in a theory is a parameter which breaks an enhanced global symmetry $\mathcal{G}$.

This definition requires a bit of an explanation. Typically we can enhance the symmetry of a theory by allowing fictitious background fields to transform under a symmetry group. Such background fields can be constructed by effectively treating a parameter as a field — when this field takes a vacuum expectation value, this spuriously enhanced symmetry will then be spontaneously broken by the VEV (which is the parameter value). This field, is then known as a **spurion field**.

Let me illustrate this with an example. Consider a single complex scalar field and a Weyl fermion with the following Lagrangian,

$$\mathcal{L} = -\int d^4x \left[\partial_\mu \phi \partial^\mu \phi^* + i\psi^\dagger \not{\partial}\psi - M_\phi^2 |\phi|^2 + \frac{1}{2} M_\psi \psi\psi\right] . \tag{1.5}$$

First note the global symmetries in the system when the mass terms are ignored. There are two global $U(1)$-symmetries, namely $\phi \mapsto e^{-i\theta'}\phi$ and $\psi \mapsto e^{-i\theta}\psi$; and a shift symmetry on the scalars, $\phi \mapsto \phi + c$ where $c$ is a constant. Here we can distinguish the spurions associated to the global symmetries. Clearly, if we set $M_\phi \to 0$ then we recover the shift symmetry; whilst if we set $M_\psi \to 0$ we will get the chiral $U(1)$-symmetry recovered — hence $M_\phi$ and $M_\psi$ are the spurions for the global shift and chiral symmetries respectively.

Why is the idea of spurions important? Turns out spurions allow us to make order-of-magnitude estimates to parameter scales in our theory. Recall that in QFT we have learnt that any field theory could be (and arguably, should be) interpreted as an effective field theory with **scale cut-off** $\Lambda$. This scale cut-off gives the scale when theory breaks down and the description of the theory is no longer valid for the energies beyond that scale. If an EFT comes from a field theory of a higher energy cut-off (like in HEP), one can now ask the question — are the parameters of the low-energy effective theory *physically natural* — i.e. do we have to choose particular parameters in my high energy theory to make certain parameters in my low-energy theory? Dirac originally considered this and came up with a very simple approach [2].

**Definition 1.2** (Dirac's naturalness principle.)**.** Suppose we have two QFTs, $S_1$ and $S_2$, with cut-offs $\Lambda_1$ and $\Lambda_2$ and $\Lambda_1 > \Lambda_2$. Then all the parameters in the low-energy theory must be at least of the order,

$$c^{(2)} \sim \mathcal{O}\left(\frac{\Lambda_2}{\Lambda_1}\right) . \tag{1.6}$$

In particular, $c^{(2)} \ll \mathcal{O}\left(\frac{\Lambda_2}{\Lambda_1}\right)$ will be unnatural.

This turns out to be too stringent of a requirement. The modern understanding of naturalness comes from 't Hooft [3] who introduced the idea of technical naturalness [2].

[2]Note that this definition is different from the modern definition of *technical naturalness*.

**Definition 1.4** ('t Hooft naturalness)**.** The effective interactions of a theory at a scale $\Lambda_2 < \Lambda_1$ should follow from the properties of a theory at a scale $\Lambda_1$, without the requirement that various parameters in the high energy theory should match at the order $\sim \Lambda_2/\Lambda_1$. In particular, if a parameter $\alpha(\Lambda_2)$ is small in the low-energy theory, setting $\alpha(\Lambda_2) = 0$ must increase the symmetry of the system.

't Hooft's idea comes from the observation that parameters can be set small if it is a near symmetry — i.e. there is some approximate symmetries that we can break weakly to obtain this small parameter. Otherwise, the small parameter must come from some careful construction of small parameters in the high-energy scale or parameters of similar scale but differ by the order $\sim \mathcal{O}(\Lambda_2/\Lambda_1)$.

Translating into spurion language, this means the spurion can be small as it is parametrises the breaking of a global symmetry. Nice. Let us now look at the consequence of naturalness in the quantum corrections of the spurion. Quantum loop order effects should correct the spurion field, and in general we will have, for $c$ a spurion,

$$c_{\text{phys}} = c_0 + \delta c \, , \tag{1.7}$$

where $c_0$ is the parameter (the free field) in the theory and $\delta c$ is the quantum corrections caused by loop effects in the theory. Let us think about what happens when we set $c \to 0$ to restore some global symmetry $\mathcal{G}$. Clearly, when we have a near symmetry, $c_{\text{phys}} \ll 1$. The bare coupling clearly contains divergences sensitive to the cut-off scale $\Lambda$ and renormalisation scheme dependences that is cancelled by $\delta c$, so we in general have

$$\delta c = f(c, \Lambda) \, . \tag{1.8}$$

At this point $f(c, \Lambda)$ can be any function we want. But we can do better. Remember what a spurion field is. A trick in spurion analysis is to promote the spurion $c$ to a field in a theory, $\tilde{c}$ which takes the parameter value as its vacuum expectation value,

$$\langle \tilde{c} \rangle = c \, . \tag{1.9}$$

What does this have to do with quantum corrections? Since setting $c \to 0$ restores the global symmetry, we must have $\delta c \to 0$ in this case. To put it more explicitly, suppose we now promote $c$ to a field $\tilde{c}$ and compute the quantum corrections to its vacuum expectation value. Then since the global (spurion) symmetry is preserved, we must have,

$$\delta \langle \tilde{c} \rangle \sim \tilde{c} \, , \tag{1.10}$$

**Definition 1.3** (Technical naturalness)**.** Parameters in a theory are called **technically natural** if their size in the UV theory is not spoiled when renormalised down to the IR at intermediate scales.

This definition is important as it is possible for parameters to be technically natural but not 't Hooft natural — for these cases an assumption is made about the absence of additional physical thresholds (corrections due to integrating out high-energy modes) intermediate between the UV and the IR which could induce large corrections in the absence of a symmetry. An example would be the super-Higgs model — supersymmetry protects the renormalisation of the Higgs mass parameter $m_H^2$, but setting $m_H^2 \to 0$ does not recover any global symmetry. The physical thresholds are absence owing to the presence of supersymmetry.

as both sides must have the same charge. In the simple model with a Weyl fermion above for example, we must have,

$$\delta M_\psi \sim M_\psi f(\Lambda, \dots) \tag{1.11}$$

where $f(\Lambda, \dots)$ must be a charge-invariant function. Why can't I add in some term which has the same charge as the spurion field $M_\psi$? Suppose we have at some energy $\Lambda_M$ where $M_\psi = 0$. Quantum corrections will now mean that at lower energies $M_\psi$ is regenerated as an operator — but $M_\psi\psi\psi$ is protected by the (spurious) chiral symmetry [3]! Therefore, $f(c, \Lambda)$ actually has the form,

$$f(c, \Lambda) = c\,(1 + \cdots)\ , \tag{1.12}$$

and have therefore arrived in the important result:

**Proposition 1.1.** *If a spurion in a theory breaks a particular symmetry, then the size of that parameter will not receive any large corrections in perturbation theory, so it is technically natural for it to be small.*

Note that here we have distinguished between a mass $M_1$ of a theory and a scale $\Lambda_1$. Very briefly said, the mass $M_1$ is where *new physics comes in*, whereas the scale $\Lambda_1$ tells you the highest energy that the theory has predictive power. So they are very different things [4].

### 1.2 The renormalised operators and the naturalness problem

Let us go back to QFT and look at operators in an effective field theory. Recall that a general effective Lagrangian is defined by the field content and the symmetries. An operator of dimension $d$ will have the coefficient in the Lagrangian,

$$\delta\mathcal{L}_{\text{eff}} \sim \frac{1}{\Lambda^{d-D}}\mathcal{O}_d \tag{1.13}$$

where $D$ is the dimension of the theory. To calculate a physical quantity, we could imagine an expansion of the form,

$$\mathcal{A} \sim \mathcal{A}_0\left[1 + \frac{M^{d-D}}{\Lambda^{d-D}} + \dots\right]\ , \tag{1.14}$$

with $M$ the kinematic scale of the physical process and $\Lambda$ the scale of the physics. We see how operators of dimension $d > D$ have less significant effects at low-energies — so we call them **irrelevant operators**. In contrast, the effects of the operators with dimension $d < D$ increase with energy so we call them **relevant**, whilst the ones with dimension $d = D$ are called **marginal** as they are independent of energy.

Let us now consider what happens to our parameters when we have relevant operators that cannot be forbidden by symmetries. Recall a model that all of you have seen in QFT (see

[3]This means in the eyes of Wilsonian EFT, any field $A$ with the same charge as $M_\psi$ will generate the coupling $A\psi\psi$ which is the same as the mass term we start with. If we wish to restore the global chiral symmetry without any spuriousness, we must set $M_\psi = A = \cdots = 0$, and this term must not be regenerated under renormalisation.

[4]The interested among you can pursue this point of view further in [4]. My discussion of spurions is loosely based of his notes as well.

Problem Sheet 4). We have a field theory consisting of a scalar with mass $\mu$ and a Dirac fermion $\psi$ of mass $m$ described by the Lagrangian density [5],

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2\right) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - g\phi\bar{\psi}\psi + \mathcal{L}_{\text{int}}(\phi) + \mathcal{L}_{\text{c.t.}}\ . \tag{1.15}$$

First let's apply spurion analysis. Notice that because of the interaction term (say $\mathcal{L}_{\text{int}}(\phi) = -\frac{\lambda}{4!}\phi^4$), there is now no global symmetry that is recovered when $\mu^2 \to 0$. We must simultaneously set $\lambda, \mu^2 \to 0$ for the shift symmetry $\phi \mapsto \phi + c$ to be restored. Therefore $-\frac{1}{2}\mu^2\phi^2$ is a renormalisable operator that are not forbidden by any global symmetry.

But consider now what happens when we carry out a one-loop calculation on $\mu^2$. The renormalised scalar mass $\mu_R^2$ can be calculated by using some regularisation scheme to obtain (schematically),

$$\mu_R^2(m) = \mu^2(m) + \frac{c_3 y^2}{16\pi^2}m^2\ , \tag{1.16}$$

where $c_3$ is a constant that depends on the regularisation scheme and we have matched the scales at $\mu = m$. Notice the dependence on the fermion mass $m$. This means that to make the scalar light compared to the mass-scale $m$, we will need to tune the renormalised couplings in the fundamental theory such that the bare mass $\mu^2$ cancels out with the term involving $m^2$ — there is no obvious symmetry principle motivating this, and we say that the scalar mass $\mu$ is fine-tuned.

This suggests the following statement.

**Definition 1.5.** The **naturalness problem for scalar mass parameters** states that the renormalised mass for scalar particles in a theory is quadratically dependent on some UV scale $\Lambda_{\text{UV}}$. The relevant operators that are not forbidden by symmetries are generally sensitive to heavy physical thresholds in the theory. This problem is regulator-independent — this dependence on UV scale will exist regardless of the regularisation scheme chosen.

To summarise, the naturalness problem arises as in general renormalisable couplings in our theory will have scale-dependent loop contributions to scalar mass parameters. This means that although the low-energy theory alone does not suffer from any problems [6], if we expect the fundamental theory to be finite and fully predictive, then these divergences mean by restricting ourselves in a subset of the full theory (the low-energy theory), we have neglected finite, physical contributions from the missing parts of the theory (in this case, the high-energy modes). This dependence on cut-off scales $\Lambda$, entirely removable by renormalisation, provides a useful proxy for the dependence on physical scales (UV-scales, $\Lambda_{\text{UV}}$) in a more UV-complete theory.

Let us go back to the theory defined in Eq.(1.15).

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2\right) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - g\phi\bar{\psi}\psi + \mathcal{L}_{\text{int}}(\phi) + \mathcal{L}_{\text{c.t.}}\ . \tag{1.15}$$

[5] I would like to thank my office mate Gaurang for enlightening me this example.

[6] By this I mean we can choose an appropriate renormalisation procedure to absorb the divergences encountered in loop calculations. You can also check that here $\mu^2$ is not technically natural in the sense defined in Definition 1.3, since there are now threshold corrections ($\sim m^2$) when we do renormalisation.

Remember in that question we were asked to calculate explicitly the counter-term for a scalar three-point function,

$$\delta\mathcal{L}_{\text{c.t.}} \ni -\frac{1}{3!}\delta_\eta \phi^3 \ . \tag{1.17}$$

Why did we need this counter-term in the first place? Notice the presence of the Yukawa term in Eq.(1.15) breaks the $\mathbb{Z}_2$-symmetry,

$$\mathbb{Z}_2 : \phi \mapsto -\phi \ , \tag{1.18}$$

and $g$ is a $\mathbb{Z}_2$-spurion. Therefore, loop corrections of parameters that breaks this symmetry must be accompanied with odd powers of $g$. In particular, after a long calculation, we will obtain,

$$\delta_\eta = -g^3 m \frac{3}{4\pi^2} \log \frac{\Lambda^2}{m^2} + f_\eta + \mathcal{O}(g^4) \tag{1.19}$$

Here, $g$ is a spurion and it is technically natural. However, because of the $\mathbb{Z}_2$-symmetry, radiatively all terms that are invariant under the spurious symmetry (by promoting $g$ to a field) $g \mapsto -g$ will be generated, in particular, the term,

$$\delta\mathcal{L} \sim \eta\phi^3 \tag{1.20}$$

will be radiatively generated with the correction given in Eq.(1.19). This now illustrates the **totalitarian principle** clearly — where anything that is not forbidden by a symmetry must be included (in this case, a spurious symmetry).

### 1.3 The hierarchy problem

We can now state the hierarchy problem.

**Definition 1.6** (The EW hierarchy problem)**.** The **electroweak hierarchy problem** is the naturalness problem that the Higgs mass being sensitive to the UV scale.

Let us illustrate this problem in a bit more detail [5]. The Standard Model can be treated as an effective field theory up to some cut-off scale $\Lambda_{\text{SM}}$. We can compute the one-loop corrections to the Higgs mass,

$$\delta m_H^2 = \frac{\Lambda_{\text{SM}}^2}{16\pi^2}\left(-6y_t^2 + \frac{9}{4}g^2 + \frac{3}{4}g'^2 + 6\lambda\right) \ . \tag{1.21}$$

If we treat the SM as an EFT with a cut-off $\Lambda_{\text{SM}}$, recall from the previous subsection that we can simply interpret these divergences as corrections that can be cancelled out by an appropriate renormalisation scheme. However, we know from our physics education that the SM is not the end of the story. Gravity is not incorporated, and when we put the SM in some more complete theory with physical scales these divergences will be replaced by finite contributions dependent on $\Lambda_{\text{UV}}$. In particular, the typical argument goes — when we set this $\Lambda_{\text{UV}} \sim M_P$ where we expect quantum gravity effects to come in, then,

$$\delta m_H^2 \sim M_P^2 \gg m_H^2 \ , \tag{1.22}$$

where $m_H^2$ is the double mass parameter inferred from the Higgs VEV $v$ and Higgs mass $m_h^2$. This is the known as the electroweak hierarchy problem, and it is by comparing scales that we know $\Lambda \sim 500$ GeV is when new physics should start to enter.

We however still don't know what the new physics is. And it is in this context that supersymmetry first came up as a plausible solution to new physics.

### 1.4 SUSY and hierarchy problem

Let us look at how SUSY saves the hierarchy problem — but we will first need to do some calculations. Recall that in the Standard Model the Higgs field $H$ is a complex scalar with a scalar potential of form,

$$V = m_H^2|H|^2 + \lambda|H|^4 \,, \tag{1.23}$$

which gives the vacuum expectation value of

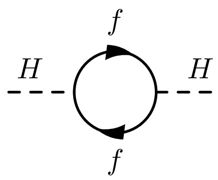
$$\langle H \rangle = \sqrt{-\frac{m_H^2}{2\lambda}} \,. \tag{1.24}$$

The problem however is that $m_H^2$ receives huge quantum corrections from virtual effects of all particle phenomenology that couples directly or indirectly to $H$, namely the term

$$\mathcal{L} \supset -\lambda_f H \bar{f} f \,. \tag{1.25}$$

Let us in particular calculate the one-loop contribution to $m_H^2$. The one-loop fermionic contribution is given by the following diagram,

$$\delta m_H^2 \quad =$$




$$+ \; \ldots \tag{1.26}$$

which is given by (using $f = t$ as the top quark gives the leading-order contributions)

$$-i\delta m_H^2|_{\text{top}} = (-1)N_c \int \frac{d^4k}{(2\pi)^4} \,\text{tr}\left[\frac{-iy_t}{\sqrt{2}}\frac{i}{\not{k} - m_t}\left(\frac{-iy_t^*}{\sqrt{2}}\frac{i}{\not{k} - m_t}\right)\right] \,. \tag{1.27}$$

If we use the hard-momentum cut-off, this term will evaluate to become,

$$\delta m_H^2|_{\text{top}} = -\frac{N_c|\lambda_t|^2}{8\pi^2}\left[\Lambda^2 - 3m_t^2 \log\left(\frac{\Lambda^2 + m_t^2}{m_t^2}\right) + \ldots\right] \,. \tag{1.28}$$

This is where SUSY comes in. Suppose we have $N$ new scalar particles $\phi_L$ and $\phi_R$, which gives the Lagrangian,

$$\mathcal{L}_{\text{scalar}} = -\frac{\lambda}{2}H^2\left(|\phi_L|^2 + |\phi_R|^2\right) - H\left(\mu_L|\phi_L|^2 + \mu_R|\phi_R|^2\right) - m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2 \,, \tag{1.29}$$

then when $N = N_c$, $\lambda = |\lambda_t|^2$ the quadratic divergences from the fermionic loop in Eq. (1.28) will be cancelled and setting $m_t = m_L = m_R$ and $\mu_L^2 = \mu_R^2 = 2\lambda m_t^2$ then also cancels out the logarithmic divergences [6]. The key is that supersymmetry seems to give a way out of this and it is a very elegant solution.

The sad thing is, with our good friend the LHC we have so far found no evidence of supersymmetry at 500GeV. In fact, the models that involve coupling supersymmetry to the Standard Model in the most simple way, known as **Minimal Supersymmetric Standard Models (MSSM)**, have now been ruled out by the LHC experiments. So to say that you should study supersymmetry because it solves the hierarchy problem is no longer, in its simplest form, valid [7].

### 1.5 The reasons why you should do supersymmetry

Okay. So we are back to square one. Not all hope is lost though. What I would really want to convince you is this — supersymmetry is still worth learning. And I have come up with three reasons.

1. **Understanding QFT better.** Throughout the course I will emphasise how much supersymmetry gives us amazing control. The reason is thus — from the QFT courses you have already seen how messy QFT are, even just on the perturbative level. This is basically all we know how to do, and when the coupling goes strong we basically are forced to stop. Supersymmetric field theories are a subset of QFTs, but they turn out to place strong restrictions on the dynamics of the theory and therefore make things tractable. They also highlight many concepts that turn out to be very useful in understanding to wider class of QFTs, such as *dualities*, *phase transitions* and more.

2. **SUSY and mathematics.** Supersymmetric theories turns out to have a lot of deep connections with mathematics. In particular, the study of topological index theorems is manifest in supersymmetric theories of low dimensions and it is in fact where many current geometry and topology research is on. If you are interested, you can have a quick look at Appendix B where I have illustrated how the localisation principle highlights this relationship with topology.

3. **SUSY and phenomenology.** Ultimately we want to describe the world. However, as we have illustrated, SUSY does not exist, at least to our best knowledge, at TeV scale where we had expected new physics to pop up. This is not the end though. There is still about 15 orders of magnitude between $M_P$ and $\Lambda_{\text{LHC}}$, and one can perhaps hope that supersymmetry will show up at some point. I am a string theorist, and if you were to believe that string theory is the best way to explain the world (as far as we know, it is the only consistent quantum gravity theory), then it seems like supersymmetry is needed for string theory. The reason is two-fold — on a historical perspective, adding fermions to string theory automatically gives you local supersymmetry (supergravity). Non-supersymmetric string theories seem to be inconsistent [8]. On a practical perspective, superstring theories are a lot better understood precisely

[7] This is the viewpoint I take. There are however still a lot of people that argue otherwise.

[8] I must admit even as a working string theorist I know very little about non-supersymmetric string theories, so I am taking the communities' word on this.

> because of the control SUSY buys us. So it might still be true that SUSY is there, and out-of-touch of whatever experiments we can construct. There are thousands of literature out there on the exact mechanisms of supersymmetry breaking (and how it related to other problems in string theory) and we will briefly discuss this later.

All hope is not lost. And I would advise you to stay with me and have fun on this little journey where we explore the many fun things about supersymmetry.

## 2 Spinor Representations

We can ask ourselves what we will need to understand supersymmetric field theories. Since supersymmetry is a symmetry between bosons and fermions, it is prudent for us to understand how to write down fermionic fields. How do we write down fermionic representations of the Lorentz group? This is the subject of this section.

### 2.1 Projective representations and spinors

In quantum mechanics, we know that physical states live in a Hilbert space. This is clearly the same in quantum field theory — a QFT is based on quantum mechanics, and the states in a QFT should be represented by a ray in a Hilbert space $\mathcal{H}$. Why a ray? The physical states in QFT are identified up to a $c$-number,

$$\Psi \sim c\Psi \ , \quad c \in \mathbb{C}^* \ , \tag{2.1}$$

and typically we normalise the state so the states are identified up to a phase.

What does this have to do with our discussion? Recall the argument in *Groups and Representations* last term — we have some symmetries in a physical theory, and we will need some way to mathematically describe them in QFT. This is done via groups and representations — symmetries are abstractly described by a group structure, and its effects on the space of states are described by its representation on that vector space. Notice that our space is set of rays in a Hilbert space — our representation space is actually not the Hilbert space, but the projectivised version on that. Instead of 'normal' or 'regular' representations, we should really look at something known as projective representations.

**Definition 2.1.** Let $G$ be a group and $V$ a finite-dimensional vector space over a field $\mathbb{F}$. A map $\rho : G \to GL(V)$ is a **projective representation** of $G$ over $\mathbb{F}$ if there exists a mapping $\alpha : G \times G \to \mathbb{F}^*$ such that the following two properties hold:

(1) $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ , $\forall x, y \in G$.

(2) $\rho(1) = \mathrm{id}_V$.

The two conditions imply that $\alpha$ satisfies the following properties:

(i) $\alpha(x, y)\alpha(xy, z) = \alpha(y, z)\alpha(x, yz)$ , $\forall x, y, z \in G$.

(ii) $\alpha(x, 1) = \alpha(1, x) = 1$ , $\forall x \in G$.

Alternatively, one can define projective representations as a map $\rho : G \to PGL(V)$, where $PGL(V)$ is the projective linear group of $V$.

Our physical Hilbert space is intrinsically a projective space, so the phase $\alpha(x, y)$ cannot be eliminated in any way. This has quite a few mathematical consequences. In deriving the Lie algebra for any symmetry, we will actually arrive at the central extension of the Lie algebra [9],

$$[T_a, T_b] = if^a{}_{bc} T_a + if_{bc} 1 \ . \tag{2.2}$$

The second term is known as the **central charge** — this modifies the Lie algebra and admits new classifications of representations. Intrinsically projective representations can arise in two ways — either via the presence of this central charge, or by the fundamental group $\pi_1(G)$ of the Lie group. We will not pursue this general line of thought here, but the interested amongst you can have a look at [7, 8].

We will however focus our attention on the Lorentz group $SO(1, 3)$. In particular, do intrinsically projective representations arise via algebra and/or topology? You probably remember from last term that we have this identity,

$$SO(1, 3) \cong \frac{SL(2, \mathbb{C})}{\mathbb{Z}_2} \ , \tag{2.3}$$

The Lorentz group is not simply connected. So it must have intrinsically projective representations. In particular, since a double loop that goes twice from 1 to $\Lambda$ to $\Lambda\tilde{\Lambda}$ and back to 1 is contractible, let us write,

$$\left[U(\Lambda)U(\tilde{\Lambda})U^{-1}(\Lambda\tilde{\Lambda})\right]^2 = 1 \ , \tag{2.4}$$

and rearranging gives,

$$U(\Lambda)U(\tilde{\Lambda}) = \pm U(\Lambda\tilde{\Lambda}) \ . \tag{2.5}$$

The same is true for the Poincaré group. This has a very important consequence — this sign identification gives two kinds of states. The states with integer spin will not be affected, by the states with half-integer spin will have a sign change when going on $2\pi$ around the axis. This gives a **superselection rule** — we do not mix states of integer and half-integer spins.

Having a superselection rule is mathematically cumbersome. Turns out there is a way we can work with regular representations. This requires the lifting the group $G$ to the central extension of the universal covering group of the classical symmetry group $\hat{G}$ — and it turns out that the projective representations will then be lifted up to regular representations of $\hat{G}$. The details are sketched out in Appendix C, and I would encourage you to have a look. The key idea, however, is to work with the central extended universal cover of $SO(1, 3)$ — i.e. $\mathrm{Spin}(1, 3) \cong SL(2, \mathbb{C})$. This is how we will construct representations of fermions — some of the representations (spinor representations) of $SL(2, \mathbb{C})$ will exactly give us what we want!

[9] This is a whole other subject and requires a lot of mathematical exposition.

## 2.2 Spinors — the physics approach

So we need fermions — the basic unit turns out to be something known as spinors. How do we construct spinors representations? For the sake of clarity, we will work with $SL(2,\mathbb{C})$ and stick with four-dimensions for the time being. We will think about how to extend these structures to other dimensions in the next subsection.

Recall that at the Lie algebra level we have the identification,

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \ , \tag{2.6}$$

so we can label the representations of $\mathfrak{sl}(2,\mathbb{C})$ with two numbers $(j_+, j_-)$. This is already treated in Groups and Representations last term [10]. The fundamental and anti-fundamental representations of $\mathfrak{sl}(2,\mathbb{C})$ are exactly the Weyl spinors, which under some $S \in SL(2,\mathbb{C})$ transform as,

$$\psi_\alpha \mapsto S_\alpha{}^\beta \psi_\beta \ , \tag{2.7}$$

$$\bar{\psi}_{\dot{\alpha}} \mapsto (S^*)_{\dot{\alpha}}{}^{\dot{\beta}} \psi_{\dot{\beta}} \ . \tag{2.8}$$

We call these **left-handed Weyl spinors** ($(j_+, j_-) = (\frac{1}{2}, 0)$) and **right-handed Weyl spinors** ($(j_+, j_-) = (0, \frac{1}{2})$) respectively. In particular, for a complex representation of a Lie group, we can find another representation by taking the conjugate, i.e. we find a matrix $C$ such that $S^* = CSC^{-1}$, so the Weyl spinors are related by,

$$(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}} \ . \tag{2.9}$$

The invariant tensors in $SL(2,\mathbb{C})$ act as invariant tensors,

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \tag{2.10}$$

with the minus signs assigned such that $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$. As discussed in the lectures, this allow us to build bilinears which properties we have checked in Q4 of the first problem sheet.

We can now construct vectors. To do this note that vector representations are the representations with $(j_+, j_-) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$. We will need the Pauli matrices,

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = (\mathbb{1}_2, \sigma^i)_{\alpha\dot{\alpha}} \ . \tag{2.11}$$

Now write a scalar related to a vector by $X = x_\mu \sigma^\mu$, then you can show that (exercise!) the spinor bilinear $\psi X \bar{\chi} = \psi^\alpha X_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$ is invariant under $SL(2,\mathbb{C})$. This means that $\psi\sigma^\mu\bar{\chi}$ is a vector. you can similarly construct $\bar{\sigma} = \epsilon\sigma^T\sigma^T$ with,

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (\mathbb{1}_2, -\sigma^i)^{\dot{\alpha}\alpha} \ . \tag{2.12}$$

and the vector is,

$$\bar{\chi}\bar{\sigma}^\mu\psi = \bar{\chi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\psi_\alpha \ . \tag{2.13}$$

[10] If you have no idea what is going on, please have a look at §5.1-5.2 of Andre's notes here.

Can we construct a different type of spinors? It turns out the representation $(j_+, j_-) = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ gives something known as **Dirac spinors**, which we can write as,

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} . \tag{2.14}$$

Here, we introduce the **Dirac matrices**,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \implies \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{2.15}$$

Here $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$ with $\sigma^i$ the Pauli matrices. We call the last line (after the arrow) a matrix representation that satisfies the **Clifford algebra**. Brushing all the mathematical details aside for now (see the following subsection), the matrix representations $\gamma^\mu$ gives the matrices,

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] , \tag{2.16}$$

which satisfies the Lorentz algebra. The vector space that these matrices act on by,

$$\psi^\alpha \mapsto S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x) , \tag{2.17}$$

is precisely the (Dirac) spinor space. We will see how we can find spinor representations in other dimensions by understanding the structure of Clifford algebras in diverse dimensions.

Back to 4d, and we now have Weyl and Dirac spinors [11]. It is important for you, in this course, to know how to operator spinor algebra. Let me summarise these rules as follows.

1. Contract undotted indices $\searrow$, $\psi^\alpha \chi_\alpha = \psi\chi$.
2. Contract dotted indices $\nearrow$, $\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi}$.
3. Treat all spinors as Grassmann-valued, $\psi^\alpha \psi^\beta = -\psi^\beta \psi^\alpha$.
4. Epsilon symmetry rules, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = \epsilon_{\beta\alpha}$.
5. Raising and lowering indices with epsilon in the front, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$ (similarly for dotted indices).
6. Contraction rule for sigmas, $(\bar{\sigma})^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}$.

You should perhaps have a go at the problem sheet again if you are unsure how to apply these rules.

[11] There are also Majorana spinors, characterised by

### 2.3 Clifford algebras and spinors

How do we extend this discussion to other dimensions? It turns out that we will need the technology of Clifford algebras. The logic is as follows. Mathematically, we will first define something known as a Clifford algebra which is an associative algebra over some field $\mathbb{K}$ satisfying the relation,

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2B_{ij}\mathbb{1} \ , \tag{2.18}$$

where $\mathbb{1}$ is the unit in the algebra. Alternatively, we can treat Clifford algebras à la Eq. (2.15), and then generate the Clifford algebras in different dimensions by tensoring up Pauli matrices. The Lorentzian versions will need some extra work. We then properly classify the Clifford algebras in different dimensions, and we will see that the spin algebra is embedded in the even-graded part of the Clifford algebra (more on that later). Looking at the representations of this subalgebra in different dimensions will then give the different spinors.

I don't want taint the main discussion of the notes with all the algebraic details. Instead, I encourage you to read Appendix D for its gory details. A physicist's way of understanding Clifford algebras is reviewed in [9], and a mathematician's approach can be found in [10, 11].

# A Example sheets feedback

## A.1 Problem sheet 1

### A.1.1 Question 1 - Poincaré symmetry

The main goal of this question is to find out how operators transform under the Poincaré group. There are some points to note:

(a) Some of you missed the fact that $M_{\mu\nu}$ is antisymmetric. Of course this just comes from the Lorentz algebra [12], but you should write,

$$\hat{M}_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) \ , \tag{A.1}$$

and not,

$$\hat{M}_{\mu\nu} = -2i(x_\mu\partial_\nu) \ . \tag{A.2}$$

If you write the last question, then you will need to also contract the indices with some antisymmetric objects $\lambda^{\mu\nu}$. Otherwise it will be incorrect.

(b) We want to check

$$U(\Lambda_1, a_1)U(\Lambda_2, a_2) = U(\Lambda_3, a_3) \ , \tag{A.3}$$

where

$$\Lambda_3 = \Lambda_1\Lambda_2 \ , \quad a_3 = \Lambda_1 a_2 + a_1 \ . \tag{A.4}$$

[12] See Andre's course on Groups and Representations.

The main point is that you need to make sure that the argument of the operator, i.e. $\mathcal{O}(x)$, also transforms appropriately. This means, you should take the active transformation carefully into account [13]. In particular note that,

$$U(\Lambda_3, a_3)^{-1}\mathcal{O}(x)^{\mathcal{A}}U(\Lambda_3, a_3) = L(\Lambda_3)^{\mathcal{A}}_{\mathcal{B}}\mathcal{O}^{\mathcal{B}}(\Lambda_3^{-1}x - \Lambda_3^{-1}a_3) \ , \tag{A.5}$$

you should check that the argument of $\mathcal{O}^{\mathcal{B}}$ matches with the one obtained by carrying out two transformations $U_1$ and $U_2$ back-to-back.

(c) This part is well done (apart from the arguments of the operators which I have already commented on). The main thing to note here is the extra term obtained,

$$[M_{\mu\nu}, \mathcal{O}^{\mathcal{A}}(x)]_{\text{new term}} = (-\mathcal{S}_{\mu\nu})^{\mathcal{A}}{}_{\mathcal{B}}\,\mathcal{O}^{\mathcal{B}}(x) \ , \tag{A.6}$$

where $\mathcal{S}_{\mu\nu}$ is a representation of the Lorentz algebra. The term exists because the field is now in a different representation space of the Poincaré group. You can think of the scalar field infinitesimal transformation as giving the relation between the field at $\Lambda^{-1}x$ versus the field at $x$. But since the field is now in a nontrivial representation there must be an extra part coming from 'representation space-contribution'.

### A.1.2 Question 2 - Clifford algebra and Lorentz generators

This is just algebra. I don't really have much to say about this. If you struggle then opening any kindergarten QFT manual should save you. In particular, the last part requires carefully writing out the indices of $\lambda^{\rho}{}_{\sigma} = \frac{i}{2}\lambda^{\mu\nu}(S_{\mu\nu})^{\rho}{}_{\sigma}$, and you will need to figure out how to write out $(S_{\mu\nu})^{\rho}{}_{\sigma}$.

### A.1.3 Question 3 - The homomorphism $SL(2,\mathbb{C}) \to SO(1,3)$

This question aims to build the homomorphism

$$\Psi : SL(2,\mathbb{C}) \to SO(1,3) \ . \tag{A.7}$$

This is sometimes known as the **spinor map**. Note that this is a two-to-one map as we will later find out that $\ker\Psi = \{\pm\mathbb{1}_2\}$. The image of the map is the identity component of $SO_{\mathbb{R}}(1,3)$, denoted typically as $SO^{+}_{1,3}(\mathbb{R})$. We then have,

$$PSL_2\mathbb{C} = \frac{SL_2\mathbb{C}}{\mathbb{Z}_2} \cong SO^{+}_{1,3}(\mathbb{R}) \ , \tag{A.8}$$

where we write $SL(2,\mathbb{C})$ as $SL_2\mathbb{C}$ and $SO(1,3)$ as $SO_{1,3}(\mathbb{R})$. Of course, since $\Psi$ is a smooth map and that $SL(2,\mathbb{C})$ is simply-disconnected, the map will always land on the identity component of $SO(1,3)$, i.e. $SO^{+}(1,3)$ or $\Lambda^{\uparrow}_{+}$ [14]. This is why there seems to be an ambiguity when I define the map $\Psi$.

Anyway, moving on to the question. Some points to note.

[13] In fact, everyone missed it.

[14] This notation is typically used to indicate the proper orthochronous Lorentz group.

- Parts (a) and (b) are generally well done. I also discussed these briefly in the class.
- Part (c) is all about spinor algebras. In particular we want to show,

$$\Lambda(A_1)^\mu{}_\nu \Lambda(A_w)^\nu{}_\rho = \Lambda(A_1 A_2)^\mu{}_\rho \ , \tag{A.9}$$

$$\Lambda(A)^\mu{}_\nu \Lambda(A)^\rho{}_\sigma \eta_{\mu\rho} = \eta_{\nu\sigma} \ . \tag{A.10}$$

  You should prove that,

$$\mathrm{Tr}\left(A_1^\dagger \bar{\sigma}^\mu A_1 \sigma_\nu\right) \mathrm{Tr}\left(A_2^\dagger \bar{\sigma}^\nu A_2 \sigma_\rho\right) \stackrel{!}{=} -2\,\mathrm{Tr}\left(A_2^\dagger A_1^\dagger \bar{\sigma}^\mu A_1 A_2 \sigma_\rho\right) \tag{A.11}$$

  for the first result and similarly for the second. This requires a bit of spinor algebra which we have developed in Q4 and in the class, and you should use prove,

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} (\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = -2\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\delta} \ . \tag{A.12}$$

  Don't try and cheat your way through — it's good practice.
- Part (d) is covered in the class — you should use the fact that since $A \in SL(2,\mathbb{C})$, this gives constraints on the matrix components. Most of you stated that $A = \pm\mathbb{1}_2$ implies $\Lambda^\mu{}_\nu = \eta^\mu{}_\nu$ which is saying that $\mathbb{Z}_2 \subset \ker\Psi$ but not the other way around. You need to show that $\pm\mathbb{1}_2$ are the only solutions to the kernel to complete the full argument by, for example, using the method mentioned in the class (i.e. set $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and imposing conditions).
- Parts (e) and (f) involve deriving infinitesimla versions of the map $\Psi$, i.e. the Lie algebra homomorphism,

$$\tilde{\Psi} : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{so}_{1,3}\mathbb{C} \ . \tag{A.13}$$

  The key point is to first derive the infinitesimal version of $\Lambda(A)^\mu{}_\nu$,

$$\lambda^\mu{}_\nu = -\frac{1}{2}\,\mathrm{Tr}\left(\delta A^\dagger \bar{\sigma}^\mu \sigma_\nu\right) - \frac{1}{2}\,\mathrm{Tr}\left(\bar{\sigma}^\mu \delta A \sigma_\nu\right) \tag{A.14}$$

  and note that this is antisymmetric [15]. Now try and prove,

$$\mathrm{Tr}\left(\delta A \sigma^{\mu\nu}\right)^* = -\,\mathrm{Tr}\left(\delta A^\dagger \bar{\sigma}^{\mu\nu}\right) \ , \tag{A.15}$$

  which results in

$$\lambda_{\mu\nu} = 2\,\mathrm{Re}\,\mathrm{Tr}\left(\delta A \sigma_{\mu\nu}\right) \ . \tag{A.16}$$

  The reverse of the map can be constructed by writing,

$$\delta A = y^{\mu\nu}\sigma_{\mu\nu} \ , \tag{A.17}$$

  and try to evaluate $\mathrm{Tr}\left(\sigma^{\mu\nu}\sigma^{\rho\sigma}\right)$ to get $\lambda_{\mu\nu} = -2y_{\mu\nu}$.

We actually haven't computed the reverse map of $\Psi$. This is in fact,

$$A = e^{i\phi}\frac{\sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu}{2\sqrt{\Lambda^\mu{}_\mu}} \tag{A.18}$$

with $\mathrm{tr}\,A = e^{i\phi}|\,\mathrm{tr}\,A|$. The phase $e^{i\phi}$ can be determined up to $\pm1$ by imposing $\det A = 1$ [16].

[15] You should check this as an exercise.

[16] For the details of this construction see Hugh Osborne's Group Theory notes, §4.3.

### A.1.4 Question 4 - Spinor algebra

This question is the most important one this sheet. You shoudl be really comfortable with the spinor algebra manipulations. The important points are the following:

1. Undotted sum goes downwards from left to right, $\psi^\alpha \chi_\alpha$.
2. Dotted sum goes upwards from left to right, $\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$.
3. $\epsilon^{\alpha\beta} = \epsilon^{-\beta\alpha} = \epsilon_{\beta\alpha}$.
4. $\left(\sigma^\mu_{\alpha\dot{\alpha}}\right)^T = \sigma^\mu_{\dot{\alpha}\alpha}$.
5. $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon^{\beta\delta} \left(\sigma^\mu\right)_{\delta\dot{\gamma}}$.
6. $\left(\sigma^\mu\right)_{\alpha\dot{\beta}} \left(\bar{\sigma}_\mu\right)^{\dot{\gamma}\delta} = -2\delta^\delta_\alpha \delta^{\dot{\gamma}}_{\dot{\beta}}$.

You should prove all of this, and then evaluate the identities in the question again (see [12] for some identities). In particular, for (c)(i), the Schouten identity is useful:

$$\epsilon_{\alpha\beta}\delta^\mu_\gamma + \epsilon_{\beta\gamma}\delta^\mu_\alpha + \epsilon_{\gamma\alpha}\delta^\mu_\beta = 0 \tag{A.19}$$

## A.2 Problem sheet 2

**Under construction**

Released after Class 2!

## A.3 Problem sheet 3

**Under construction**

Released after Class 3!

## A.4 Problem sheet 4

**Under construction**

Released after Class 4!

# B Localisation

In this section we sketch out two properties of supersymmetric Lagrangians in zero dimensions.

- **Localisation.** Partition function localises around critical points of the superpotential in a supersymmetric theory.
- **Deformation Invariance.** Partition function is invariant under the change in the potential.

We will sketch out these two ideas in more detail. We will see how supersymmetric QFTs have a special property where the partition function localises to specific points in the functional space and how this is related to topological quantities. This section is mainly based on David Skinner's SUSY notes [17] and [13].

### B.1 Localisation

The idea of localisation is simple - in a supersymmetric theory, the value of the relevant path integral reduces to a much smaller-dimensional integral. In some cases this reduces to counting contributions of certain points in the field space.

Let us illustrate this in the zero-dimensional case. Using Berezin integration rules, where

$$\int d\psi = 0 \; , \quad \int \psi d\psi = 1 \; , \tag{B.1}$$

the simplest form of a non-trivial action is of the form [18],

$$S(X, \psi_1, \psi_2) = S_0(X) - \psi^1\psi^2 S_1(X) \; . \tag{B.2}$$

The partition function $Z$, for which we define as,

$$Z = \int \prod_i dX^i \prod_a d\psi^a e^{-S(X,\psi)} \; , \tag{B.3}$$

is then,

$$Z = \int dX e^{-S_0} S_1(X) \; , \tag{B.4}$$

using the Berezin integration rules. What is the simplest case for a supersymmetric transformation to exist? We can define a real function called the **superpotential** [19] $W : \mathcal{F} \to \mathbb{R}$ where $\mathcal{F}$ is the space of functions $x = X$ and defining, $\psi = \psi^1 + i\psi^2$ and $\bar{\psi}$ to be the conjugate variable let us write,

$$S_0(x) = \frac{1}{2}(\partial W(x))^2 \; , \tag{B.5}$$

$$S_1(x) = \partial^2 W(x) \; , \tag{B.6}$$

so

$$S(X, \psi, \bar{\psi}) = \frac{1}{2}(\partial W)^2 - \psi\bar{\psi}\partial^2 W \; . \tag{B.7}$$

Then the action $S$ is invariant under the flow generated by the fermionic vector fields,

$$\mathcal{Q} = \psi\frac{\partial}{\partial x} + \partial W(x)\frac{\partial}{\partial \bar{\psi}} \; , \tag{B.8}$$

$$\mathcal{Q}^\dagger = \psi\frac{\partial}{\partial x} - \partial W(x)\frac{\partial}{\partial \bar{\psi}} \; , \tag{B.9}$$

[17] The notes by David Skinner in Cambridge is where I have learnt a lot of mathematical physics from - they are really good and it would be a shame if you give them a miss!

[18] We will need at least two fermionic variables as the action is in the even algebra and $\psi^2 = 0$.

[19] We call this the superpotential for nomenclature reasons - this will become clear when we look at other theories.

with the nontrivial transformations being,

$$\mathcal{Q}(x) = \psi \ , \tag{B.10}$$

$$\mathcal{Q}(\bar{\psi}) = \partial W(x) \ , \tag{B.11}$$

and similarly for $\mathcal{Q}^\dagger$ [20]. These vector fields are exactly odd derivations of $C^\infty(\mathbb{R}^{1|2})$ and are the supercharges that generate supersymmetries of this zero-dimensional theory. Looking at the anticommutators, we see that,

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\partial W(x)\psi\frac{\partial}{\partial\bar{\psi}} \ , \quad \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = -2\partial W(x)\bar{\psi}\frac{\partial}{\partial\psi} \ , \tag{B.12}$$

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = -\partial W(x)\left(\psi\frac{\partial}{\partial\psi} - \bar{\psi}\frac{\partial}{\partial\bar{\psi}}\right) \ . \tag{B.13}$$

It is a bit weird to analyse the supersymmetric algebra here but here we note two things - firstly, the RHS of Eq. (B.13) shouldn't be interpreted as the Hamiltonian - we don't have time in zero-dimensions. The second point concerns with Eq. (B.12) - we see indeed $\mathcal{Q}^2$ is not zero in general but since $\psi\partial^2 W(x) = 0$ is the equation of motion the supersymmetric algebra indeed vanishes on-shell [21].

Localisation from coordinate transformations.

Let us first try and understand localisation from a coordinate transformation perspective. In particular, we would like to evaluate the path integral,

$$\mathcal{Z} = \int e^{-S} dx d^2\psi \ . \tag{B.14}$$

Firstly let us isolate the neighbourhoods $\mathcal{U}$ where the derivative superpotential $\partial W$ vanishes. Taking the complement $\mathcal{U}^c$ of $\mathcal{U}$ in $\mathcal{F}$, we can change variables $(x, \psi, \bar{\psi}) \mapsto (y, \chi, \bar{\chi})$ where,

$$y = x - \frac{\psi\bar{\psi}}{\partial W} \ , \quad \chi = \psi\sqrt{\partial W} \ , \quad \bar{\chi} = \bar{\psi} \ . \tag{B.15}$$

The new measure is now,

$$dx d^2\psi = \sqrt{\partial W(y)} dy d^2\chi \ , \tag{B.16}$$

where $\mathcal{Q}(y) = 0 = \mathcal{Q}^\dagger(y)$ so $y$ is invariant under supersymmetry [22]. We also see that the action transforms as,

$$S[y, 0, 0] = \frac{1}{2}\left(\partial W(y)\right)^2 = S[x, \psi, \bar{\psi}] \ . \tag{B.17}$$

[20] But with $\mathcal{Q}^\dagger(\psi) = -\delta W(x)$.

[21] This is the similar to the case in the free Wess-Zumino model hen we missed out degrees of freedom. Turns out we have simply missed out a bosonic auxiliary field - if we include such contribution, such as using the superfield formalism in $\mathbb{R}^{0|2}$, this will allow us to reproduce the full supersymmetry algebra with $\{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = \{\mathcal{Q}, \mathcal{Q}^\dagger\} = 0$.

[22] In fact $y$ is the only independent combination of $(x, \psi, \bar{\psi})$ that is supersymmetrically invariant so any invariant function will be a function of $y$.

The contribution to the path integral is surprisingly,

$$\mathcal{Z}_{\mathcal{U}^c} = \frac{1}{2\pi}\int_{\mathcal{U}^c} e^{-S[y,0,0]}\sqrt{\partial W(y)}dyd^2\chi = 0\ , \tag{B.18}$$

due to the property of the Berezin integral. This means that the non-vanishing contributions to $\mathcal{Z}$ only comes from the neighbourhood $\mathcal{U}$ - this is exactly where the coordinate transformation breaks down as $\partial W \to 0$ means the Jacobian of the coordinate transformation is no longer invertible. This leads us to the following key observation.

**Proposition B.1** (Localisation principle.)**.** *Quantum field theories with supersymmetry generically have path integrals that localise to a vicinity of a fixed point set.*

How do we further evaluate this? Consider the case where $W$ is a generic polynomial of degree $d$ with $d-1$ isolated non-degenerate [23] critical points. Then around this critical point $x = x^*$, we can write,

$$W(x) = W(x^*) + \frac{\alpha_c}{2}(x-x^*)^2 + \dots\ , \tag{B.19}$$

with $\alpha_c = \partial^2 W(x^*)$. Then the action becomes,

$$S(x,\psi,\bar{\psi}) = \frac{\alpha_c^2}{2}(x-x^*)^2 - \alpha_c\bar{\psi}\psi\ , \tag{B.20}$$

and expanding the exponential in Grassmann variables in the integral will yield,

$$\begin{aligned}
\mathcal{Z} &= \sum_{x^*}\frac{1}{\sqrt{2\pi}}\int dxd^2\psi e^{-\frac{1}{2}\alpha_c^2(x-x^*)^2}\left(-1+\alpha_c\bar{\psi}\psi\right) \\
&= \sum_{x^*}\frac{\alpha_c}{\sqrt{2\pi}}\int e^{-\frac{1}{2}\alpha_c^2(x-x^*)^2} \\
&= \sum_{x^*}\frac{\alpha_c}{|\alpha_c|}\ ,
\end{aligned} \tag{B.21}$$

which eventually leads to,

$$\mathcal{Z} = \sum_{x^*:\partial W|_{x^*}=0}\frac{\partial^2 W(x^*)}{|\partial^2 W(x^*)|} \tag{B.22}$$

This is a surprising result. We note that if $d$ is odd then $\mathcal{Z}=0$, and if $d$ is even then $\mathcal{Z}=\pm 1$ as we have $d-1$ critical points. $\mathcal{Z}$ just counts the number of times the superpotential crosses $W=0$ [24]!

There is perhaps another way to illustrate this result. To do this we will need to discuss something know as deformation invariance.

[23] This means $\partial^2 W|_{x^*} \neq 0$.

[24] Or if you like, the number of kinks as a one-dimensional instanton.

### B.2 Deformation Invariance

Deformation invariance can be summarised in one sentence: the path integral $\mathcal{Z}$ is sensitive only to the order of polynomial in $W$.

What do I mean by that? Let's suppose a quantum field theory has some symmetry $G$ where it leaves the action and path integral measure invariant. Then the correlation functions of quantities that are variables of fields under the symmetry vanishes. To see this, let's suppose $g$ is a field, and $f$ is defined as the variation of $\phi$ under symmetry $G$,

$$f = \delta_G \phi \ , \tag{B.23}$$

Then the expectation value of $f$ is,

$$\langle f \rangle = \int f e^{-S} = \int \delta_G g e^{-S} = \int \delta \left( g e^{-S} \right) = 0 \ . \tag{B.24}$$

For the present case, we can set,

$$g = \partial \rho(X) \bar{\psi} \ . \tag{B.25}$$

Now the variation of $g$ under supersymmetry gives,

$$f = \epsilon \left( \partial \rho \partial W - \partial^2 W \psi \bar{\psi} \right) \ , \tag{B.26}$$

which leaves

$$\left\langle \partial \rho \partial W - \partial^2 W \psi \bar{\psi} \right\rangle = 0 \ . \tag{B.27}$$

Now since the action is,

$$S(X, \psi, \bar{\psi}) = \frac{1}{2} (\partial W)^2 - \psi \bar{\psi} \partial^2 W \ , \tag{B.7}$$

we can see that Eq. (B.27) gives the invariance of the correlation function,

$$\langle \delta_\rho S \rangle = 0 \ , \tag{B.28}$$

under the transformation of the superpotential $W \mapsto W + \rho$. This shows that the partition function is invariant under a change in the potential - which is true as long as $\rho$ is small at infinity in field space when compared to $h$ so the boundary terms in the argument will indeed vanish [25]. In particular, we can rescale $W \mapsto \lambda h$, with $\lambda \gg 1$. Then as long as,

$$\partial W e^{-\lambda^2 (\partial W)^2 / 2} \to 0 \tag{B.29}$$

when $|x| \to \infty$, boundary terms will not appear and the partition function $\mathcal{Z}$ will remain invariant. In particular, looking at the path integral $\mathcal{Z}$ now defined with this deformation parameter $\lambda$,

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int dx d^2 \psi e^{-S_\lambda} \ , \tag{B.30}$$

[25] $\rho$ can be of the same order as $h$ as long as the leading order term is smaller than that of $h$.

where the action is now,

$$S_\lambda(X,\psi,\bar{\psi}) = \frac{\lambda^2}{2}(\partial W)^2 - \lambda\psi\bar{\psi}\partial^2 W \ , \tag{B.31}$$

we see that, given the limit in Eq. (B.29) we will have,

$$\frac{d}{d\lambda}\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}}\int dx d^2\psi \mathcal{Q}_\lambda^\dagger \left(\psi\partial W e^{-S_\lambda}\right) \ , \tag{B.32}$$

which gives zero as no boundary terms will survive. The deformation invariance of path integral allows us to deduce that for $\lambda \to 0$, $e^{-\lambda^2(\partial W)^2/2}$ suppresses all contributions to the integral arbitrarily strongly apart from around the neighbourhood $\mathcal{U}$ of points $x^*$ : $\partial W(x^*) = 0$. This is the other way to understand the localisation principle.

The deformation principle allow us to consider deformations of the superpotential $W(x)$. In particular, if $W(x)$ is a polynomial of order $d$, then we can deform $W(x)$ such that it has no critical points if $d$ is odd and only one critical point if $d$ is even - this is the same crossing phenomenon we have commented in the previous section, and we shall later see how this generalises to topological formulae in higher dimensions.

### B.3 Explicit evalution

In fact, it is possible to evaluate directly the path integral $\mathcal{Z}$ in Eq. (B.14). We can write,

$$\begin{aligned}
\mathcal{Z} &= \frac{1}{\sqrt{2\pi}}\int dx d^2\psi e^{-S} \\
&= \frac{1}{\sqrt{2\pi}}\int dx \partial^2 W e^{-\frac{1}{2}(\partial W)^2} \\
&= \frac{D}{\sqrt{2\pi}}\int dy y e^{-\frac{1}{2}y^2} \\
&= D
\end{aligned} \tag{B.33}$$

where $D$ is the degree of the map $x \mapsto y = \partial W(x)$. It enters the equation as the map is not one-to-one. The degree counts the number of preimages of a given point taking into account the relative orientation of each preimage with respect to its image so $D$ is 0 and $\pm 1$ respectively in the cases where $d$ is odd and even, exactly as before.

One more side comment. A 'third' way of understanding localisation is to interpret the fermionic symmetry as some symmetry acting on the path integral $G$. In the most general case when $G$ is freely acting, the integral over $G$ just factors out (c.f. integration over an orbit in group theory like the Haar measure). The relevant integral here is $\int_G d\theta = 0$. However, our $G$, the group of fermionic symmetries parametrised by fermionic coordinate $\theta$, has fixed locus $\mathcal{C}_0$ precisely in the open neighbourhood $\mathcal{U} \subset \mathcal{C}$ where $\mathcal{C} = \mathcal{F}$ is the space over which the integral is performed. Localisation exactly comes from these fixed points where the coordinate transformation is not well-defined as this is the fixed point of the fermionic symmetry $G$ (generated by $\mathcal{Q}^\dagger$).

## C Projective representations and covers

In the main text we have discussed how one should consider projective representations in QFTs. The central argument there is that instead of considering the projective representation, we can lift the group to the universal cover of the group and look at its 'normal' representations.

To properly discuss this requires a bit of topology set-up — in this section I will aim to provide a comprehensive account on what is needed to understand this theorem.

### C.1 Covers and universal covers

We begin by defining what a covering map is [14].

**Definition C.1.** A subset $U \subset X$ is **evenly covered** by $\pi$ if $U$ is connected and open, and each component $\pi^{-1}(U)$ is an open set that is mapped homeomorphically onto $U$ by $\pi$.

**Definition C.2.** A **covering map** is a continuous surjective map $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is path-connected and locally path-connected[26], and every point $p \in X$ has an evenly covered neighbourhood. We call $\tilde{X}$ the covering space of $X$ and $X$ the base of the covering.

Of course, everything so far is in the topology context. To specialise this to smooth manifolds (which is what we want), we will need to restrict the definition to a very specific type of covering map[27].

**Definition C.3.** Take $E$ and $M$ connected smooth manifolds with or without boundary. A map $\pi : E \to M$ is called a **smooth covering map** if $\pi$ is smooth and surjective, and each point in $M$ has a neighbourhood $U$ such that each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto $U$ by $\pi$. We say $U$ is evenly covered. We call $M$ to be the **base manifold**, and $E$ a **covering manifold of** $M$. If $E$ is simply-connected, it is called the **universal covering manifold of** $M$.

Here *simply-connected* means every loop is path-isomorphic to a constant path [28]. We want to show that this universal covering exists and is in fact unique. I will here quote a few lemmas and theorems without detailed proof - the details can be found in the references [14, 16].

**Theorem C.1.** *Suppose $M$ is a connected smooth-$n$-manifold, and $\pi : E \to M$ is a topological covering map. Then $E$ is a topological $n$-manifold, and has a unique smooth structure such that $\pi$ is a smooth covering map.*

[26] It might be surprising to see how path-connectedness does not generally imply locally path-connectedness. A counterexample is the topologist's sine curve, $y = \sin\left(\frac{1}{x}\right)$ for $x \in (0, \pi)$ together with closed arc connecting $(0, 0)$ and $(\pi, 0)$ where the space is path-connected but not locally path-connected at $(0, 0)$.

[27] By the way, if you are completely baffled by the definitions I just made, these are just mathematical details that you can skip (if you want, but I am weird so I will babble on). Alternatively you should pick up some topology books and start learning what topology is.

[28] Phrased in the language of the fundamental group at $X$, simply-connectedness simply means that the fundamental group of a manifold at every point $q \in M$ is the trivial group [14, 15].

*Proof.* See Proposition 4.40 of [14]. ☐

**Corollary C.1.** *If $M$ is a connected smooth manifold, there exists a simply connected manifold $\tilde{M}$ - the* ***universal covering manifold of*** *$M$, and a smooth covering map $\pi : \tilde{M} \to M$. The universal covering manifold is unique such that for any other universal covering manifold $\tilde{M}'$ with projection map $\pi'$, then there exists a diffeomorphism $\Phi : \tilde{M} \to \tilde{M}'$ such that $\pi' \circ \Phi = \pi$.*

*Proof.* This is Corollary 4.43 of [14]. Since a proof is not given there I will give a sketch of the proof. The first step is show that any connected and locally simply connected space admits a unique universal cover. You will need some sort of path-connectedness arguments (see Theorem 12.8 of [16]) - to show the path classes are lifted to the upper space, then check several topological requirements (path-connectedness, topologies, covering maps). Now by Theorem C.1 you have the existence of a smooth covering manifold of $M$ that is simply-connected. To show uniqueness, we need to find $\phi$ between any two universal covers that is a diffeomorphism - to show this find open sets such that you can find a surjective smooth submersion of $\pi\big|_{V^{-1}}$ must give you a smooth $\phi$ and $\phi^{-1}$ in both directions. ☐

Now it is straightforward to generalise this to Lie groups.

**Theorem C.2.** *Let $G$ be a connected Lie group. There exists a simply connected Lie group $\tilde{G}$, called the universal covering group of $G$, that admits a smooth covering map $\pi : \tilde{G} \to G$ that is also a Lie group homomorphism.*

*Proof.* See Theorem 7.7 of [14]. Essentially the idea is you now need to also do group axiom checks on the universal covering group. ☐

**Theorem C.3.** *For any connected Lie group $G$, the universal covering group is unique in the following sense: if $\tilde{G}$ and $\tilde{G}'$ are connected Lie groups with corresponding smooth covering maps $\pi$ and $\pi'$, then there exists a Lie group homomorphism $\Phi : \tilde{G} \to \tilde{G}'$ such that $\pi' \circ \Phi = \pi$.*

*Proof.* Again - this is similar to the proofs done above. See Theorem 7.9 of [14]. ☐

### C.2 Projective representations

In the main text we have defined projective representations of $G$ as the map $\rho : G \to PGL(V)$. Let me expand a bit on the definition stated above and make things a bit more precise. Recall in quantum mechanics we need states to be positive-definite to have a notion of probability in the Hilbert space — operators are therefore unitary or anti-unitary by Wigner's theorem. Then, we have the following definition.

**Definition C.4.** Let $U(V)$ be the group of invertible linear transformations of a finite-dimensional Hilbert space $V$ over $\mathbb{C}$ that preserve the inner product. A finite-dimensional **unitary representation** of a matrix Lie group $G$ is a continuous homomorphism of $\Pi : G \to U(V)$ for some finite-dimensional Hilbert space $V$.

**Definition C.5.** Let $V$ be a finite-dimensional Hilbert space over $\mathbb{C}$. The **projective unitary group** over $V$, denoted $PU(V)$ is then the quotient group

$$PU(V) = U(V)/e^{i\theta}I \tag{C.1}$$

where $e^{i\theta}I$ denotes the group of matrices in $U(1)I$, $I$ being the identity matrix.

This establishes the codomain of the representation map. In particular, it can be shown that $PU(V)$ is isomorphic to a matrix Lie group[29]. Now let $Q : U(V) \to PU(V)$ be the quotient homomorphism and let $q : \mathfrak{u}(V) \to \mathfrak{pu}(V)$ be the associated Lie algebra isomorphism. We note that given an ordinary unitary representation $\Sigma : G \to U(V)$, we can always form a projective representation $\Pi : G \to PU(V)$ by setting $\Pi = Q \circ \Sigma$. This is equivalent to saying the following diagram commutes:

$$\begin{tikzcd} & G \arrow[dl, "\Sigma"'] \arrow[dr, "\Pi"] & \\ U(V) \arrow[rr, "Q"] & & PU(V) \end{tikzcd}$$

Note that not all projective representations arise in this fashion. I will state the following propositions without detailed proof.

**Proposition C.1.** *If $V$ is a finite-dimensional Hilbert space over $\mathbb{C}$, then $PU(V)$ is isomorphic to a matrix Lie group. The associated Lie algebra homomorphism $q$ defined above has the kernel $\{iaI\}$, so $PU(V)$ is isomorphic to $U(V)/\{iaI\}$.*

*Proof.* Consider the homomorphism $\Gamma : U(V) \to GL(gl(V))$, such that for given $U \in U(V)$, $\Gamma : U \mapsto C_U(X) = UXU^{-1}$. Then one can show that $\ker \Gamma = \{U(1)I\}$, so the image under this homomorphism is isomorphic to the quotient group $U(V)/\{e^{i\theta}I\}$, compact, and closed, i.e. a matrix Lie group isomorphic to $PU(V)$. To find the related Lie algebra homomorphism, we note that $c_X(Y) = [X, Y]$, with the kernel of $c_X$ being the scalar multiples of $I$ in $U(V)$ - the group $\{iaI\}$. The map $c_X$ therefore must map onto $PU(V)$, giving the required isomorphism. $\square$

Every finite-dimensional projective representation can be "de-projectivised" at the Lie-algebra level. To state this we have the following proposition.

**Proposition C.2.** *Let $\Pi : G \to PU(V)$ be a finite-dimensional projective unitary representation of a matrix Lie group $G$, and $\pi : \mathfrak{g} \to PU(V)$ be the associated Lie algebra homomorphism. Then there exists a Lie algebra homomorphism $\sigma : \mathfrak{g} \to U(V)$ such that $\pi(X) = q(\sigma(X)) \quad \forall X \in \mathfrak{g}$. So the following diagram commutes:*

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ U(V) \arrow[rr, "q"] & & PU(V) \end{tikzcd}$$

[29] See Proposition 16.44 of [17].

*This $\sigma$ is unique upon fixing that* $\operatorname{tr}\sigma(X) = 0 \quad \forall X \in \mathfrak{g}$.

*Proof.* This proposition boils down to the fact that you can always fix $\sigma(X)$ to have trace zero by choosing for $Y \in \mathfrak{u}(1)$, pick $\sigma(X) = Y + cI$ where $c$ is an appropriate pure-imaginary constant. Such $\sigma$ therefore always exist. (See Proposition 16.46 of [17] for more details.) □

Now we can say the most important theorem in this subsection:

**Theorem C.4.** *Suppose $G$ is a matrix Lie group and $\tilde{G}$ is a universal cover of $G$ with the covering map $\Phi$. Then the following hold:*

1. *Let $\Pi : G \to PU(V)$ be a finite-dimensional projective unitary representation of $G$. Then there is an ordinary unitary representation $\Sigma : \tilde{G} \to U(V)$ of $\tilde{G}$ such that $\Pi \circ \Phi = Q \circ \Sigma$. Any such $\Sigma$ is irreducible if and only if $\Pi$ is irreducible. $\Sigma$ is unique if we choose* $\det(\Sigma(A)) = 1,\ A \in \tilde{G}$.

2. *Let $\Sigma$ be a finite-dimensional irreducible unitary representation of $\tilde{G}$. Then the kernel of the associated projective unitary representation $Q \circ \Sigma$ contains the kernel of the covering map $\Phi$. Therefore $Q \circ \Sigma$ factors through $G$ and gives rise to a projective unitary representation of $G$.*

*Point 1 is equivalent to saying that the following box diagram commutes:*

$$\begin{tikzcd} \tilde{G} \arrow[r, "\Pi"] \arrow[d, "\Phi"'] & U(V) \arrow[d, "Q"] \\ G \arrow[r, "\Sigma"] & PU(V) \end{tikzcd}$$

*Proof.* See Theorem 16.47 of [17]. The idea is actually really simple - we make use of Proposition C.2 to find an ordinary representation of $\mathfrak{g}$ at the base level, and then simply lift it up and apply Lie's Theorem at the cover level. The second half the theorem rests on the fact that $\ker\Phi$ is a discrete normal subgroup $\tilde{G}$ and is therefore central. We can then show that $\Sigma(A)$, where $A \in \ker\Phi$ under Schur's lemma gives $\Sigma(A) = cI$ as it intertwines $V$ to itself. The $A$ is in the kernel of the associated projective representation $Q \circ \Sigma$. □

Of course, we should be dealing with the infinite-dimensional case. Here of course the unitary representation needs to be defined slightly different[30] The main thing to note here is that we can no longer do the de-projectivisation by passing to the Lie algebra since there is no unique member we can choose - the notion of trace doesn't work for unbounded operators on the Hilbert space. Point 1 in Theorem **??** no longer works. However, if $G$ is connected and "semi-simple", every projective unitary representation of $G$ can be de-projectivised after passing to the universal cover. This is in fact the crucial reason why

[30] You will need some sort of a strong continuity homomorphism $\Phi : G \to U(\boldsymbol{H})$. You can read more about this in [17].

we need to study the universal covering manifolds! The spins intrinsically comes from this universal cover, and it is precisely since we are looking at the de-projectivised version of the representation that brought us there in the first place!

### C.3 Central extensions

We are still not done. Recall the precisely statement we made in §2 is that there is a lifting of unitary representation of some projective Hilbert space, $\rho : G \to U(\mathbb{P}(\mathcal{H}))$ to the *central extension* of the universal covering group of the classical symmetry group. We haven't discussed anything about central extensions so far!

So let us properly address what we mean by the central extension of the universal covering group.

**Definition C.6.** An **extension of $G$ by the group $A$** is given by an exact sequence of group homomorphisms,

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1 \, . \tag{C.2}$$

The extension is **central** if $A$ is abelian and its image $\mathrm{im}(\iota)$ is in the centre of $E$, i.e.

$$a \in A \;\; b \in E \Longrightarrow \iota(a)b = b\iota(a) \tag{C.3}$$

Let us illustrate this by examples.

**Example C.1.** A **trivial extension** has the form,

$$1 \to A \xrightarrow{i} A \times G \xrightarrow{pr_2} G \to 1 \, . \tag{C.4}$$

A non-trivial example is the following,

$$1 \to \mathbb{Z}/k\mathbb{Z} \to E = U(1) \xrightarrow{\pi} U(1) \to 1 \, . \tag{C.5}$$

**Example C.2** (Semi-direct products)**.** Recall that the **semidirect product** $G \ltimes H$ with a homomorphism $\tau : G \to \mathrm{Aut}(H)$ is given by,

$$(g, h) \cdot (g', h') = (gg', h\tau(g)(h')) \, . \tag{C.6}$$

This can be written as the extension,

$$1 \to H \xrightarrow{\iota} G \ltimes H \xrightarrow{\pi} G \to 1 \, , \tag{C.7}$$

with $\pi(g, h) = h$ and $\iota(h) = (a, h)$ for fixed $a \in G$. For example, the semidirect group $GL(V) \ltimes V$ is isomorphic to the group of affine transformations.

**Example C.3** (Lorentz group)**.** Obviously we can write,

$$1 \to \mathbb{Z}_2 \to SL(2, \mathbb{C}) \to SO(1, 3) \to 1 \, , \tag{C.8}$$

where $\pi$ is the two-to-one covering.

**Example C.4.** For a vector space $V$ over field $\mathbb{F}$, we have,

$$1 \to \mathbb{F}^* \xrightarrow{\iota} GL(V) \xrightarrow{\pi} PGL(V) \to 1\ , \tag{C.9}$$

with $\iota : \lambda \mapsto \lambda \mathrm{id}_V$. Then $PGL(V)$ is the projective linear group which we have used in the definition of the projective representations.

Why do we need to set this up? Suppose $\mathcal{H}$ is a Hilbert space and $\mathbb{P}(\mathcal{H})$ is the projective space of 1d linear subspaces of $\mathcal{H}$, then we will have the central extension,

$$1 \to U(1) \xrightarrow{\iota} U(\mathcal{H}) \xrightarrow{\hat{\gamma}} U(\mathbb{P}) \to 1\ . \tag{C.10}$$

From Wigner's theorem [31], we know that for every projective transformation $T \in \mathrm{Aut}(\mathbb{P})$ which are set of all projective transformations that preserve the transition probability, there exists a unitary or an anti-unitary operator $U \in U(\mathcal{H})$ where $T = \hat{\gamma}(U)$. So it is nice — we indeed have all the notions that we had before but now in the projective setting. But we could also ask: Given a projective representation $T$ such that there is a continuous homomorphism $T : G \to U(\mathbb{P})$, does there exist a unitary representation $S : G \to U(\mathcal{H})$, such that the following diagram

$$\begin{tikzcd}
 & & & G \arrow[dl, "\rho"'] \arrow[d, "\rho_P"] & \\
1 \arrow[r] & U(1) \arrow[r] & U(\mathcal{H}) \arrow[r, "\hat{\gamma}"'] & U(\mathbb{P}) \arrow[r] & 1
\end{tikzcd}$$

commutes? Note that this is different from the above Wigner's theorem as this is not about the automorphisms of the projective Hilbert space $\mathbb{P}(\mathcal{H})$ but about representations! This turns out to be in general not achievable, and the key statement is the following.

**Proposition C.3.** *Given a representation $\rho_P : G \to U(\mathbb{P})$, there exists a lifting with respect to the central extension of the universal covering group of the classical symmetry group, $\tilde{\rho} : \hat{G} \to U(\mathcal{H})$.*

*Proof.* I will not discuss the proof and its glory details here, instead you can find the details in Theorem 3.10 in [8] which gives a natural lift of the representation by a $U(1)$ extension, and then use Bargmann's Theorem which states that every connected and simply-connected $G$ admits such a lift to finish the proof. □

The short form is the following. Since the universal cover of $G$ is just the central extension by the fundamental group $\pi_1(G)$, we can always apply Proposition C.3 to work with the universal cover and apply Bargmann's Theorem. This is merely a restatement of the result in Theorem C.4 — the results don't change, but merely highlights the need to work with universal covering groups in physics so we have a chance of working with $U(\mathcal{H})$. Phew!

[31] See the Appendix of [8] for the proof.

# D Clifford algebras and spinors

This section deals with Clifford algebras and spinor representations properly. It is quite mathematically heavy, so please look away if you find maths horrifying.

## D.1 Real, imaginary and quaternionic representations

Before we begin let us address some notions about representations. Fix $G$ a group.

**Definition D.1.** Let $V$ be a vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. A **Hermitian form** on $V$ is an $\mathbb{F}$-bilinear form $V \times V \to \mathbb{F}$ such that for all $v_1, v_2 \in V$ and $\lambda \in \mathbb{F}$,

$$\langle v_1, \lambda v_2 \rangle = \lambda \langle v_1, v_2 \rangle \ , \quad \langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle} \ . \tag{D.1}$$

Suppose $G$ is a finite group , a compact Lie group or a semisimple Lie group. Then $V$ has a $G$-invariant non-degenerate hermitian form. If $|G| = \infty$ the proof is a bit more complicated. It turns out that Clifford algebras is almost the group algebra of a finite group, so very similar arguments apply there.

**Definition D.2.** Let $V$ be a complex vector space. A linear map $\varphi : V \to V$ is a **real (quaternionic)** structure if $\varphi$ obeys the following conditions:

1. $\varphi$ is conjugate linear, $\varphi(\lambda v) = \bar{\lambda}\varphi(v)$ for $\lambda \in \mathbb{C}$ and $v \in V$,
2. $\varphi^2 = 1$ ($-1$ in the quaternionic case).

We then have the following lemma.

**Lemma D.1.** *Let $V$ be a complex vector space and $c : V \to V$ be a real structure. Then $V = V_+ \oplus V_-$ where $V_\pm$ are isomorphic real vectors spaces, or $V \cong \mathbb{C} \otimes V_+$.*

*Proof.* $c^2 = 1$ implies the eigenvalues of $c$ are $\pm 1$; and since $c$ is conjugate linear $V_\pm$ are real subspaces. The isomorphism map is given by $i : V_+ \to V_-$. □

The real structure is simply a notion of complex conjugation. A quaternionic structure however allows us to define a left action $\mathbb{H}$ on $V$. In particular, suppose $J : V \to V$ is a quaternionic structure, then $J^2 = -1$ and $-iJ = Ji$, so if $q = a + bj$ is a quaternion then $qv = av + bJ(v)$ for $v \in V$.

**Definition D.3.** Let $V$ be a complex representation of $G$. We say that $V$ is of **real (quaternionic)** type if $V$ processes a $G$-invariant real (quaternionic) structure.

**Theorem D.1.** *A complex representation $V$ of $G$ is of real (quaternionic) type iff $V$ admits a non-degenerate symmetric (anti-symmetric) $G$-invariant complex bilinear form $B : V \times V \to \mathbb{C}$.*

*Proof.* The proof is a bit complicated and requires a bit of set-up. Let $B : V \times V \to \mathbb{C}$ a non-degenerate complex bilinear $G$-invariant map that satisfy $B(v_1, v_2) = \epsilon B(v_2, v_1)$ with $\epsilon = \pm 1$. With a $G$-invariant hermitian form $\langle \cdot, \cdot \rangle$ on $V$, we can define,

$$B(v_1, v_2) = \langle \varphi(v_1), v_2 \rangle \ , \tag{D.2}$$

for some $\varphi : V \to V$. This is conjugate linear, $G$-invariant and an isomorphism. It can be shown that [18] $\varphi$ can be rescaled to give the required structure map. Conversely, given the structure map $\mathcal{J} : V \to V$ satisfying $\mathcal{J}^2 = \epsilon \mathbb{1}$, we can take the $G$-invariant symmetric non-degenerate $\mathbb{R}$-bilinear form and extend by complex linearity to a non-degenerate $G$-invariant symmetric $\mathbb{C}$-bilinear form on $V$. The rest of the details are in [18]. $\square$

Let us illustrate this using some basis. Firstly, we can write some Hermitian form using a basis to a Hermitian matrix $A$,

$$\langle u, v \rangle = \bar{u}^T A v \ . \tag{D.3}$$

Let $g$ be the matrix representing the group action of $g \in G$. Then, the $G$-invariance of the Hermitian form means,

$$\bar{g}^T \cdot A \cdot g = A \ . \tag{D.4}$$

Similarly, a bilinear form can be represented by a matrix $B$,

$$B(u, v) = u^T B v \ , \tag{D.5}$$

and $G$-invariance demands,

$$g^T \cdot B \cdot g = B \ . \tag{D.6}$$

Let us now illustrate how real and quaternionic structures can be represented. For $V \cong \mathbb{C}^n$ a complex vector space, $B$ a non-degenerate $\mathbb{C}$-bilinear form can be represented by $B^T = \epsilon B$. Now the structure map $\mathcal{J} : V \to V$ is conjugate linear so can only be represented by some real basis. So we look at the underlying real vector space with a complex structure $I : V_{\mathbb{R}} \to V_{\mathbb{R}}$, given by

$$I_{\mathbb{R}} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \tag{D.7}$$

and the matrices $A$ and $B$ are represented by,

$$A_{\mathbb{R}} = \begin{pmatrix} A & -iA \\ iA & A \end{pmatrix} \ , \quad B_{\mathbb{R}} = \begin{pmatrix} B & iB \\ iB & -B \end{pmatrix} \ . \tag{D.8}$$

Now since we want $B(u, v) = \langle \mathcal{J}(u), v \rangle$. We can take the conjugate to get $\epsilon \overline{B(v, u)} = \langle v, \mathcal{J}(u) \rangle$, and in a basis we will find that,

$$A_{\mathbb{R}} \cdot J_{\mathbb{R}} = \epsilon \bar{B}_{\mathbb{R}} \ . \tag{D.9}$$

Since $A_{\mathbb{R}}$ and $B_{\mathbb{R}}$ are invertible, this constrains $J_{\mathbb{R}}$ and we will get at the end,

$$J_{\mathbb{R}} = \epsilon A_{\mathbb{R}}^{-1} \bar{B}_{\mathbb{R}} \ . \tag{D.10}$$

## D.2 Clifford Algebras and Gamma Matrices

Let us now construct Clifford algebras. The following discussion mainly follows [19].

### D.2.1 Clifford algberas, abstractly

We begin with a definition. In this section we fix $V$ to be a vector space and $B : V \times V \to \mathbb{K}$ to be a symmetric bilinear form on $V$ (here $\mathbb{K}$ is a field). To construct a corresponding quadratic form $Q$, we can define

$$Q(x) = B(x, x) \; , \tag{D.11}$$

such that $(V, Q)$ is a **quadratic vector space over** $\mathbb{K}$. One can conversely reconstruct the symmetric bilinear form $B$ from $Q$ by polarisation, i.e.

$$B(x, y) = \frac{1}{2} \left( Q(x + y) - Q(x) - Q(y) \right) \; , \tag{D.12}$$

so we hereby denote $Q$ and $B$ interchangeably, writing $Q(x) = Q(x, x)$. Let us also define what an associative algebra is.

**Definition D.4.** An **associative algebra over** $R$ is a ring $A$ together with a ring homomorphism from $R$ into the centre of $A$, i.e. for $r \in R$ and $a, b \in A$ then,

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y) \; . \tag{D.13}$$

Now we have the following definition. I have here chosen the sign conventions to conform with [10, 11, 18].

**Definition D.5.** Let $A$ be an associative $\mathbb{K}$-algebra and $(V, Q)$ be a quadratic vector space. A $\mathbb{K}$-linear map $\phi : V \to A$ is **Clifford** if $\forall x \in V$,

$$\phi(x)^2 = -Q(x) 1_A \; , \tag{D.14}$$

where $1_A$ is the unit of $A$.

**Definition D.6.** The **Clifford algebra** $C = C(Q) = \mathrm{Cliff}(V, Q)$ is an associative algebra with unit 1 and is generated by $V$ such that for all $v \in V$ we have,

$$v \cdot v = -Q(v, v) \cdot 1 \; . \tag{D.15}$$

Equivalently (if the characteristic of $k$ is not 2), we have $\forall v, w \in V$,

$$v \cdot w + w \cdot v = -2Q(v, w) \; . \tag{D.16}$$

A note about construction. The Clifford algebra can be constructed quickly by taking the tensor algebra,

$$T^{\bullet}(V) = \bigoplus_{n \geq 0} V^{\otimes n} \; , \tag{D.17}$$

and setting

$$C(Q) = \frac{T^\bullet(V)}{I(Q)} \ . \tag{D.18}$$

Here $I(Q)$ is the two-sided ideal generated by all elements of the form $v \otimes v - Q(v,v) \cdot 1$. Clearly $C(Q)$ satisfies the universal property. From this we can see that the dimension of $C$ is $2^m$ where $m = \dim(V)$ and that the canonical mapping $V \to C$ is an embedding, with the basis of $C(Q)$ being the products $e_I = e_{i_1} \cdot e_{i_2} \cdot ... \cdot e_{i_k}$ where $e_i$ are the basis of $V$. To see this, in particular, we can check the following.

**Proposition D.1.** *There is a natural embedding $V \hookrightarrow C(Q)$ which is the image of $V = V^{\otimes 1}$ under the canonical projection*

$$\pi_q : T^\bullet(V) \to C(Q) \ , \tag{D.19}$$

*and this is an injection.*

*Proof.* Say that an element $\varphi \in T^\bullet(V)$ is of pure degree $s$ if $\varphi \in V^{\otimes s}$. We want to show that any element $\varphi \in T^\bullet(V) \cap V$ is zero. Suppose this is not true. Then we can write $\varphi = \sum_i a_i \otimes (v_i \otimes v_i + Q(v_i)) \otimes b_i$ where we assume that $a_i$ and $b_i$ is of pure degree. Now since $\varphi \in V$ we must have that the expression is equal to zero, with the sum taken over those indices with $\deg a_i + \deg b_i$ maximal. Contracting with Q means $\sum_i a_i Q(v_i) \cdot b_i = 0$. Proceed with induction to show $\varphi = 0$. $\square$

The Clifford Algebra has a universal property as follows. This also gives a categorical definition of Clifford Algebras.

**Proposition D.2.** *The Clifford algebra can be defined to be the universal algebra with the following property: If $A$ is any associative algebra with unit and a linear mapping $j : V \to A$ is given such that*

$$j(v) \cdot j(v) = -Q(v,v) \cdot 1, \quad \forall v \in V \ , \tag{D.20}$$

*or equivalently $\forall v, w \in V$, (given that $k$ has a characteristic not equal to 2,)*

$$j(v) \cdot j(w) + j(w) \cdot j(v) = -2Q(v,w) \cdot 1 \ . \tag{D.21}$$

*then there should be a unique homomorphism of algebras from $C(Q)$ to $A$ extending $j$, i.e. $j$ extends uniquely to a $\mathbb{K}$-algebra homomorphism $\tilde{j} : C(Q) \to A$, and $C(Q)$ is the unique associative $\mathbb{K}$-algebra with this property.*

*Proof.* Any linear map $j : V \to A$ extends to a unique algebra homomorphism $\bar{j} : T^\bullet(V) \to A$. Now Eq. (D.20) implies that $\bar{j} = 0$ on $I(Q)$ so $\bar{j}$ descends to $C(Q)$. Suppose now $B$ is an associative $\mathbb{K}$-algebra with unit and that $\iota : V \to B$ is an embedding with the property that any linear map $j : V \to A$ with the property in Eq. (D.20) extends uniquely to an algebra homomorphism $\tilde{j} : A \to B$. Then the isomorphism from $V \subset Q$ to $\iota(V) \subset B$ clearly induces an algebra isomorphism $C(Q) \xrightarrow{\cong} B$. $\square$

The proposition above effectively states the following. Given an associative algebra with unit $A$, together wiht a Clifford map $i : V \to C(Q)$ such that for every Clifford map $\phi : V \to A$ there is a unique algebra morphism $\Phi : C(Q) \to A$ that makes the following triangle commute.

$$\begin{tikzcd}
& V \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\
C(Q) \arrow[rr, "\Phi"] & & A
\end{tikzcd}$$

Categorically, the Clifford Algebra is an initial object in the category $\mathbf{Cliff}(V, Q)$, which has Clifford maps $\phi : V \to \cdot$ from a fixed vector space equipped with a quadratic form $Q$ as objects. The morphism from $V \to A$ to $V \to A'$ is given by a commuting triangle

$$\begin{tikzcd}
& V \arrow[dl] \arrow[dr] & \\
A \arrow[rr, "f"] & & A'
\end{tikzcd}$$

with $f : A \to A'$ as a homomorphism of associative algebras. This initial object is unique up to a unique isomorphism. In other words, the Clifford algebra $C(Q)$ is universal for Clifford maps to associative algebras. The construction via tensor algebra as before implies the following statement. If $\phi : V \to A$ is a Clifford map and $\tilde{\Phi} : T^\bullet(V) \to A$ is the unique extension of $\phi$ to the tensor algebra, then $\tilde{\Phi}$ indeed annihilates the ideal $I(Q)$ and therefore factors through a unique map $\Phi : T^\bullet(V)/I(Q) \to A$ from the quotient. Therefore, we have a commutative diagram:

$$\begin{tikzcd}
V \arrow[r] \arrow[dr, "i"'] \arrow[drr, dotted] & T^\bullet(V) \arrow[d, "\phi"] \arrow[dr, "\tilde{\Phi}"] & \\
& C(Q) \arrow[r] & A
\end{tikzcd}$$

Here $i$ is really injective as the ideal only comes into play for $V^{\geq \otimes 2}$.

### D.2.2 Constructing Clifford algebras

The way we have been discussing about Clifford algebras is not very suitable for computations. Instead, we will discuss the way that Clifford introduced the algebras. This is the way Clifford algebras are still taught in physics courses, following Dirac.

Traditionally, the discussion of Clifford algebras started with Dirac matrices.

**Definition D.7.** Suppose $\{e_i\}$ is a $\mathbb{K}$-basis for $V$, where $i = 1, ..., \dim V$. The vector space $V$ is equipped with the symmetric bilinear form where $B(e_i, e_j) = B_{ij} = B_{ji}$. The **Clifford generators** $\Gamma_i$ is the image of $e_i$ under the map $i : V \to C(Q)$, which satisfy the relations

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2B_{ij}\mathbb{1} \tag{D.22}$$

where $\mathbb{1}$ is the unit in the Clifford algebra $C(Q)$.

Following this, we can define Clifford algebras by using generators in the following manner.

**Definition D.8** (Clifford algebras — generators)**.** An associative algebra over field $\mathbb{K}$ with unity 1 is the **Clifford algebra** $C(Q)$ of a non-degenerate quadratic form $Q$ on $V$ if it contains $V$ and $\mathbb{K} = \mathbb{K} \cdot 1$ as distinct subspaces such that the following three conditions hold:

(i) $v^2 = Q(v)$ for any $v \in V$.

(ii) $V$ generates $C(Q)$ as an algebra over $\mathbb{K}$.

(iii) $C(Q)$ is not generated by any proper subspace of $V$.

We can immediately how the Dirac matrices furnishes a representation of the Clifford algebra. We then define

$$\Gamma_{ij} = \frac{1}{2}\left(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i\right) \ , \tag{D.23}$$

as the product of two generators. More generally, we have

$$\Gamma_{i_1 \ldots i_p} = \frac{1}{p!}\sum_{\sigma\in S_p}(-1)^{\sigma}\Gamma_{i_{\sigma(1)}}\ldots\Gamma_{i_{\sigma(p)}} \ , \tag{D.24}$$

where $(-1)^{\sigma}$ indicates the sign of the permutation in $S_p$. We then see that since $C(Q)$ is generated by $V$ and the identity it is the linear span of $\mathbb{1}$, $\Gamma_i$, $\Gamma_{ij}$,... in total there are $1 + n + C_2^n + \ldots C_n^n = 2^n$ monomials. So $\dim C(Q) = 2^{\dim V}$. This is the same dimension as the exterior algebra $\mathrm{Ext}^{\bullet} V$ so we can establish a vector space isomorphism between the two.

In particular, if we use an orthonormal basis to generate $C(Q)$, then the first condition in the above Definition D.8 then becomes

$$\Gamma_i^2 = 1, \quad 1 \le i \le p, \tag{D.25}$$

$$\Gamma_i^2 = -1, \quad p < i \le p, \tag{D.26}$$

$$\Gamma_i\Gamma_j = -\Gamma_j\Gamma_i, \quad i < j. \tag{D.27}$$

whilst condition (iii) becomes

$$\Gamma_1 \ldots \Gamma_n \neq \pm 1 \ . \tag{D.28}$$

This is important in constructing a representation of Clifford algebras in general dimensions. Typically, this is needed in the discussion of supersymmetry and supergravity (and spinors) in various dimensions. The construction typically involves a set of matrices called **Dirac matrices** or **Gamma matrices**, defined as matrix representations of the Clifford algebra in various dimensions. You should have seen **Pauli matrices** in your elementary quantum mechanics courses:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{D.29}$$

These matrices can generate a basis for Clifford algebras of arbitrary dimensions. Here we follow the discussion in [9, 20, 21]. We construct the Euclidean $\gamma$-matrices from Gamma matrices which are the basic building block of the matrix representations of the Clifford algebras. We define $(2k+1)$-matrices by the tensor products of $k$ Pauli matrices to get a $2^k \times 2^k$ matrix representation as follows:

$$\Gamma_1^{(k)} = \sigma_1 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-1} , \quad \Gamma_2^{(k)} = \sigma_2 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-1} ,$$
$$\Gamma_3^{(k)} = \sigma_3 \otimes \sigma_1 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-2} , \quad \Gamma_4^{(k)} = \sigma_3 \otimes \sigma_2 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-2} ,$$
$$\vdots$$
$$\Gamma_{2k-1}^{(k)} = \underbrace{\sigma_3 \otimes ... \otimes \sigma_3}_{k-1} \otimes \sigma_1 , \quad \Gamma_{2k}^{(k)} = \underbrace{\sigma_3 \otimes ... \otimes \sigma_3}_{k-1} \otimes \sigma_2 ,$$
$$\Gamma_{2k+1}^{(k)} = \underbrace{\sigma_3 \otimes ... \otimes \sigma_3}_{k} \tag{D.30}$$

The matrices listed above can be generated using the recurring relations:

$$\Gamma_M^{(k+1)} = \Sigma_M^{(k)} \otimes \sigma_0 , \quad M = 1, ..., 2k \tag{D.31}$$

$$\Gamma_{2k+i}^{(k+1)} = \Sigma_{2k+1}^{(k)} \otimes \sigma_i , \quad i = 1, 2, 3 \tag{D.32}$$

which gives

$$\left\{ \Gamma_M^{(k)}, \Gamma_N^{(k)} \right\} = -2\delta_{MN} \tag{D.33}$$

So then we have the following definition.

**Definition D.9** (Gamma matrices)**.** The **Gamma** or **Dirac matrices** are matrix representations of the Clifford algebras, i.e. the map: $\Gamma : C(Q) \to \mathrm{GL}(\mathbb{C}^{2^k})$ where we map the generators $e_M \mapsto \Gamma_M$. The representation is faithful when $d = 2k$ and non-faithful when $d = 2k+1$ where $\Gamma(\epsilon) = \Gamma_1^{(k)} ... \Gamma_{2k+1}^{(k)} = i^k$.

We will find Gamma matrices extremely helpful later when we construct spinors in spaces of Euclidean and Lorentzian signatures. In particular, for Lorentzian gamma matrices we will need to do some modifications. To obtain these, we pick some single matrix from the Euclidean construction, multiply by $i$ and label it as $\Gamma_0$. This matrix is anti-Hermitian and satisfies,

$$\Gamma_0^2 = -\mathbb{1} . \tag{D.34}$$

We relabel the remaining set of gamma matrices to obtain the other gamma matrices. Then the Hermitian properties of the Lorentzian gamma matrices are then given by,

$$\Gamma_M^\dagger = \Gamma_0 \Gamma_M \Gamma_0 . \tag{D.35}$$

There is a caveat — these gamma matrices are identified up to a conjugacy class,

$$\Gamma_M \sim S \Gamma_M S^{-1} , \tag{D.36}$$

in particular, we can choose a unique irrep which is Hermitian for even dimensions and for odd dimensions there are two mathematically inequivalent irreducible representations which differ only by the sign of the final gamma matrix $\Gamma_{2N+1}$.

### D.2.3 $\mathbb{Z}_2$-grading and exterior algebras

Let us return to the tensor construction of the Clifford algebras. Since the ideal $I(Q)$ is not homogeneous, $C(Q)$ does not inherit a $\mathbb{Z}$-grading from $T^\bullet(V)$. However, notice that the ideal $I(Q)$ is generated by elements of an even degree. This means the Clifford algebra does inherit a $\mathbb{Z}_2$ grading. To study this grading recall the following definitions from elementary algebra.

**Definition D.10.** A **graded ring** is a ring that is decomposed into a direct sum of additive groups

$$R = \bigoplus_{n=0}^{\infty} R_n = R_0 \oplus R_1 \oplus R_2 \oplus \dots \ , \tag{D.37}$$

such that

$$R_m R_n \subseteq R_{m+n} \ , \tag{D.38}$$

for all non-negative integers $m$ and $n$.

**Definition D.11.** An associative algebra $A$ over a ring $R$ is **graded** if it is graded as a ring.

So we can now go back to Clifford Algebras. Consider the automorphism $\alpha : C(Q) \to C(Q)$ which sends $\alpha(v) = -v$ on $V$. Since $\alpha^2 = \mathrm{id}$, the ideal $I(Q)$ is generated by elements of an even degree, and hence Clifford algebra inherits a $\mathbb{Z}_2$ grading:

$$C(Q) = C^0(Q) \oplus C^1(Q) \ , \tag{D.39}$$

where $C^i(Q) = \left\{\varphi \in C(Q) \,\middle|\, \alpha(\varphi) = (-1)^i \varphi\right\}$ are the eigenspaces of $\alpha$. Since $\alpha$ is a homomorphism, we have

$$C^i(Q) \cdot C^j(Q) \subset C^{i+j}(Q) \ , \tag{D.40}$$

with the indices taken modulo 2. This $\mathbb{Z}_2$-grading plays an important role in the analysis and application of Clifford algebras. In particular, $C^0(Q)$ is often called $C^{\text{even}}(Q)$ and is a subalgebra of dimension $2^{m-1}$, where as $C^1(Q)$ is often called $C^{\text{odd}}(Q)$.

The $\mathbb{Z}_2$-gradedness of the Clifford algebra is very different from the graded nature of the tensor algebra which inherently has a $\mathbb{Z}$-graded structure. To see this, define $\tilde{\mathcal{F}}$ as

$$\tilde{\mathcal{F}}^r = \sum_{s \le r} V^{\otimes s} \ . \tag{D.41}$$

This has the property

$$\tilde{\mathcal{F}}^r \otimes \tilde{\mathcal{F}}^s \subset \tilde{\mathcal{F}}^{r+s} \ . \tag{D.42}$$

The tensor algebra therefore has a natural filtration

$$\tilde{\mathcal{F}}^0 \subset \tilde{\mathcal{F}}^1 \subset ... \subset T^\bullet(V) \ , \tag{D.43}$$

which makes the tensor algebra into a **filtered algebra**. Every filtered algebra has an associated graded algebra. For the tensor algebra with the canonical filtration described above, the associated graded algebra is described by

$$\tilde{\mathcal{G}}^p = \tilde{\mathcal{F}}^p / \tilde{\mathcal{F}}^{p-1} \, . \tag{D.44}$$

Then $\tilde{\mathcal{G}}^\bullet$ is a graded algebra where the product map is defined by

$$\tilde{\mathcal{G}}^p \times \tilde{\mathcal{G}}^q \to \tilde{\mathcal{G}}^{p+q} \, . \tag{D.45}$$

The canonical filtration of the tensor algebra $T^\bullet(V)$ defines a natural filtration on the Clifford algebra $C(Q)$. Suppose $\pi_q : T^\bullet(V) \to T^\bullet(V)/I(Q)$ where $I(Q)$ is the ideal that generates the Clifford algebras. Then $\mathcal{F}^i = \pi_q(\tilde{\mathcal{F}}^i)$ naturally has a natural filtration,

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \ldots \, , \tag{D.46}$$

and naturally the associated graded algebra $\mathcal{G}^r = \mathcal{F}^r/\mathcal{F}^{r-1}$ naturally inherits the filtration. We now have the following proposition.

**Proposition D.3.** *For any quadratic form $Q$, the associated graded algebra of $C(Q)$ is naturally isomorphic to the exterior algebra* $\mathrm{Ext}^\bullet V$*.*

*Proof.* The map $\bigotimes^r V \xrightarrow{\pi_r} \mathcal{F}^r \to \mathcal{G}^r = \mathcal{F}^r/\mathcal{F}^{r-1}$ given by $v_{i_1} \otimes \ldots \otimes v_{i_r} \mapsto [v_{i_1} \ldots v_{i_r}]$ descends to a map $\mathrm{Ext}^r V \to \mathcal{F}^r$ by the property in Eq. (D.21). (Note that when the characteristic of $\mathbb{K}$ is 2 then we will have to use the other condition.) This map is surjective and gives a homomorphism of graded algebras $\mathrm{Ext}^\bullet V \to \mathcal{G}^\bullet$. It remains to check the map is injective. The kernel of $\bigotimes^r V \to \mathcal{G}^r$ consists of the $r$-homogeneous pieces of elements $\varphi \in I_q(V)$ of degree less than $r$. Any such $\varphi$ can be written as a finite sum $\varphi = \sum a_i \otimes (v_i \otimes v_i + q(v_i)) \otimes b_i$ where $v_i \in V$ and where we may assume that the $a_i$ and $b_i$ are of pure degree with $\deg a_i + \deg b_i \leq r-2$. The $r$-homogeneous part of $\varphi$ is then of the form $\varphi_r = \sum a_i \otimes v_i \otimes v_i \otimes v_i$ where $\deg a_i + \deg b_i = r-2$ for each $i$. The image of $\varphi$ in the exterior algebra is however zero as $v_i \wedge v_i = 0$. So the map $\mathrm{Ext}^r V \to \mathcal{G}^r$ is injective. $\square$

Note that the proposition above gives a canonical vector space isomorphism that is compatible with the filtrations as follows,

$$\mathrm{Ext}^\bullet V \to C(Q) \, . \tag{D.47}$$

The map in Eq. (D.47) is of course not an isomorphism of algebras unless $q = 0$. However the map is indeed canonical so we can discuss embeddings of the form $\mathrm{Ext}^r V \subset C(Q)$ for all $r \geq 0$. To see that the isomorphism is only true when $q = 0$, consider the $\mathbb{Z}_2$-grading on the tensor algebra defined with $T^\bullet(V) = T^\bullet(V)_0 + T^\bullet(V)_1$ where

$$T^\bullet(V)_0 = \bigoplus_{k\geq 0} V^{\otimes 2k} \, , \quad T^\bullet(V)_1 = \bigoplus_{k\geq 0} V^{\otimes 2k+1} \, . \tag{D.48}$$

where the $\mathbb{Z}_2$-grading is the reduction mod-2 of the $\mathbb{Z}$-grading of the tensor algebra as discussed above. This reduction makes the ideal $I_q$ homogeneous, and hence the projection

$T^\bullet(V) \to C(Q)$ restricts to projections $TV_i \to C_i$ for $i = 0, 1$. Note however that for $i = 1$ this is only a projection of vector spaces, since neither $TV_1$ nor $C_1$ are algebras.

Now the canonical filtration of the tensor algebra $T^\bullet(V)$ defines a filtration on $C(Q)$ as follows. By filtering $T^\bullet(V)_0$ and $T^\bullet(V)_1$ separately, i.e.

$$\mathcal{F}^{2k}T(V)_0 = \bigoplus_{l \le k} V^{\otimes 2l} \; , \quad , \mathcal{F}^{2k+1}T(V)_1 = \bigoplus_{l \le k} V^{\otimes 2l+1} \tag{D.49}$$

such that

$$0 \subset \mathcal{F}^0 T(V)_0 \subset \mathcal{F}^2 T(V)_0 \subset \ldots \; , \tag{D.50}$$
$$0 \subset \mathcal{F}^1 T(V)_1 \subset \mathcal{F}^3 T(V)_1 \subset \ldots \; . \tag{D.51}$$

Now under the projections $TV_0 \to C_0$ and $TV_1 \to C_1$, we can similarly identify the filtrations of the Clifford algebra as

$$0 \subset \mathcal{F}^0 C_0 \subset \mathcal{F}^2 C_0 \subset \ldots \; , \tag{D.52}$$
$$0 \subset \mathcal{F}^1 C_1 \subset \mathcal{F}^3 C_1 \subset \ldots \; . \tag{D.53}$$

We will henceforth use the shorthand $\mathcal{F}^p C$ as $\mathcal{F}^p C_0$ and $\mathcal{F}^p C_1$ if $p$ is even and odd respectively. Now we note that $\mathcal{F}^p C / \mathcal{F}^{p-2} C \cong \operatorname{Ext}^p V$ as the corrections in replacing $xy$ by $-yx$ where $x, y \in V$ involve terms of degree 2 less. The corrections are 0 when $q = 0$, so we can identify $C(Q) \cong \operatorname{Ext}^\bullet V$, exactly as mentioned above.

It is possible to understand the relation between the Clifford and exterior algebras in a different way which does not involve filtrations. The bilinear form $B$ defines a map $\flat : V \to V^*$ where $x \mapsto B(x, \cdot)$. The map $flat$ is an isomorphism if and only if $B$ is non-degenerate. The inverse is typically defined as $\sharp$ so together with the map $\flat$ they are referred to as the musical isomorphisms induced from the inner product $B$. We can then define a linear map $\phi : V \to \operatorname{End}(\operatorname{Ext}^\bullet V)$ by

$$\phi(x)\alpha = x \wedge \alpha \iota_x \alpha \tag{D.54}$$

where $\iota_x$ is the unique odd derivation defined by $\iota_x 1 = 0$ and $\iota_x y = B(x, y)$ for $y \in V$. So on a monomial we have,

$$\iota_x \left( y_1 \wedge \ldots \wedge y_p \right) = \sum_{i=1}^{p} (-1)^{i-1} B(x, y_i) y_1 \wedge \ldots \wedge \hat{y_i} \wedge \ldots \wedge y_p \; , \tag{D.55}$$

where the hat denotes omission. Then we can extend this linearly to all of $\operatorname{Ext}^\bullet V$ as in the following lemma.

**Lemma D.2.** *The map $\phi : V \to \operatorname{End} V$ in Eq. (D.54) is Clifford.*

*Proof.* For every $x \in V$ and $\alpha \in \operatorname{End} \operatorname{Ext}^\bullet V$, we have

$$\begin{aligned} \phi(x)^2 \alpha &= \phi(x) \left( x \wedge \alpha - \iota_x \alpha \right) \\ &= x \wedge x \wedge \alpha - x \wedge \iota_x \alpha - Q(x)\alpha + x \wedge \iota_x \alpha + \iota_x \iota_x \alpha \\ &= -Q(x)\alpha \; , \end{aligned} \tag{D.56}$$

where $x \wedge x = 0 = \iota_x \iota_x$ and $\iota_x(x \wedge \alpha) = Q(x)\alpha - x \wedge \iota_x \alpha$. □

By the universality of Clifford algebras we can then extend this to the algebra homomorphism uniquely,

$$\Phi : C(Q) \to \operatorname{End}\operatorname{Ext}^\bullet V \ . \tag{D.57}$$

So composing this with the evaluation at $1 \in \operatorname{Ext}^\bullet V$ gives a linear map $\Phi_1 : C(Q) \to \operatorname{Ext}^\bullet V$. This map obeys $\Phi_1(1) = 1$ and if $x \in V$ then $\Phi_1(i(x)) = x$ where $i : V \to C(Q)$. Since $i$ is injective from the construction of $C(Q)$, $\Phi_1 \circ i$ is also injective. By further computations, we then get

$$\Phi_1(i(x)i(y)) = x \wedge y - B(x, y) \ , \tag{D.58}$$

and

$$\Phi_1(i(x)i(y)i(z)) = x \wedge y \wedge z - B(x, y)z + B(x, z)y - B(y, z)x \ , \tag{D.59}$$

so $\Phi_1$ surjects onto $\operatorname{Ext}^\bullet V$. This is a vector space isomorphism with the inverse map defined by

$$y_1 \wedge ... \wedge y_p \mapsto \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma y_{i_{\sigma(1)}} \wedge ... \wedge y_{i_{\sigma(p)}} \ , \tag{D.60}$$

which gives an explicit quantisation of the exterior algebra.

### D.2.4 Clifford algebras as representations of $\mathfrak{so}_n\mathbb{C}$

It is clear from the construction that Clifford algebras are associative algebras. As a results it determines a Lie algebra with the bracket defined by the associative multiplication. How are Clifford algebras related to the representations of $\mathfrak{so}_m\mathbb{C}$? To see this we will first embed the Lie algebra $\mathfrak{so}(Q)$ inside the Lie algebra of the even part of the Clifford algebra and from there identify $C(Q)$ with one or two copies of matrix algebras.

Let us see how this works in practice. We first need to make an explicit isomorphism of $\operatorname{Ext}^2 V$ with $\mathfrak{so}(Q)$, which is defined as

$$\mathfrak{so}(Q) = \{X \in \operatorname{End}(V) \,|\, Q(Xv, w) = -Q(v, Xw) \ \forall v, w \in V\} \ . \tag{D.61}$$

Define the map

$$\varphi_{a\wedge b}(v) = 2\left(Q(b, v)a - Q(a, v)b\right) \ , \tag{D.62}$$

which gives the isomorphism $\phi : \operatorname{Ext}^2 V \to \mathfrak{so}(Q) \subset \operatorname{End}(V)$ with $a \wedge b \mapsto \varphi_{a\wedge b}$. One can check that the bracket on $\operatorname{Ext}^2 V$ makes this an isomorphism of Lie algebras with the Clifford algebra[32], allowing the map $\psi : \operatorname{Ext}^2 V \to C(V, Q)$ to be defined by

$$\psi(a \wedge b) = \frac{1}{2}(a \cdot b - b \cdot a) = a \cdot b - Q(a, b) \ . \tag{D.63}$$

This is an injective embedding, which shows that the following Lemma.

**Lemma D.3.** *The mapping $\psi \circ \varphi^{-1} : \mathfrak{so}(Q) \to C(Q)^{even}$ embeds $\mathfrak{so}(Q)$ as a Lie subalgebra of $C(Q)^{even}$.*

[32] This is done by checking the brackets on $[a \wedge b, c \wedge d]$ and $[a \cdot b, c \cdot d]$.

*Proof.* See discussion above. $\square$

The reason why the embedding only goes into the even part is because, simply, $C(Q)^{\text{odd}}$ is indeed not an algebra. You can also see that in Eq. (D.63) the map is defined with elements of even degree. By looking at the basis elements we can then see that $\psi$ is an embedding and the map exactly maps the exterior algebra to the even part of the Clifford algebra.

What remains is to identify the subalgebra of $C(Q)_0$ or $C(Q)^{\text{even}}$, the image of $\mathfrak{so}(Q)$ as matrix algebras. Let us separate this into two cases.

**Case 1: $n = \dim V$ is even.**

We first decompose $V$ into two $n$-dimensional isotropic spaces for $Q$,

$$V = W \oplus W' \ . \tag{D.64}$$

Then we have the following lemma.

**Lemma D.4.** *The decomposition $V = W \oplus W'$ determines an isomorphism of algebras,*

$$C(Q) \cong \mathrm{End}(\mathrm{Ext}^\bullet W) \tag{D.65}$$

*where* $\mathrm{Ext}^\bullet W = \mathrm{Ext}^0 W \oplus ... \oplus \mathrm{Ext}^n W$.

*Proof.* Let us try and construct the map $\varphi : C(Q) \to E = \mathrm{End}(\mathrm{Ext}^\bullet W)$. The map $\varphi$ is the same as defining a linearly mapping $V \to E$ with the condition in Eq. (D.21). We must therefore construct maps $l : W \to E$ and $l' : W' \to E$ such that

$$l(w)^2 = 0 = l'(w')^2 \tag{D.66}$$

$$l(w) \circ l'(w') + l'(w') \circ l(w) = 2Q(w, w')I \ , \tag{D.67}$$

for any $w \in W$, $w' \in W$. For each $w \in W$, let $L_w \in E$ be the left multiplication by $w$ on the exterior algebra $\mathrm{Ext}^\bullet W$,

$$L_w(\xi) = w \wedge \xi, \quad \xi \in \mathrm{Ext}^\bullet W \ . \tag{D.68}$$

For any $\vartheta \in W^*$, let $D_\vartheta \in E$ be the derivation of $\mathrm{Ext}^\bullet W$ such that,

$$D_\vartheta(1) = 0 \tag{D.69}$$

$$D_\vartheta(w) = \vartheta \in \mathrm{Ext}^0 W = \mathbb{C} \tag{D.70}$$

$$D_\vartheta(\zeta \wedge \xi) = D_\vartheta\zeta \wedge \xi + (-1)^{\deg(\zeta)}\zeta \wedge D_\vartheta(\xi) \ , \tag{D.71}$$

where $w \in W = \mathrm{Ext}^1 W$. i.e. Explicitly,

$$D_\vartheta(w_1 \wedge ... \wedge w_r) = \sum_i (-1)^{i-1}\vartheta(w_i)\,(w_1 \wedge ... \wedge \hat{w}_i \wedge ... \wedge w_r) \ . \tag{D.72}$$

Now we can set

$$l(w) = L_w \ , \quad l'(w') = D_\vartheta \ , \tag{D.73}$$

where $\vartheta \in W^*$ is defined by $\vartheta(w) = 2Q(w, w')$, $\quad \forall w \in W$. It is straightforward to show that the maps defined obeys the requirements, as well as for $\zeta \wedge \xi$ if they obey for $\zeta$ and $\xi$ separately. The map is clearly an isomorphism and one can see that by its action of a basis. $\square$

Now note that there exists a decomposition of the exterior powers into even and odd parts $\mathrm{Ext}^{\bullet} W = \mathrm{Ext}^{\mathrm{even}} W \oplus \mathrm{Ext}^{\mathrm{odd}} W$ where $C(W)^{\mathrm{even}}$ respects the splitting. From Lemma D.4, we then have the isomorphism,

$$C(Q)^{\mathrm{even}} \cong \mathrm{End}(\mathrm{Ext}^{\mathrm{even}} W) \oplus \mathrm{End}(\mathrm{Ext}^{\mathrm{odd}} W) \; . \tag{D.74}$$

Combining this with Lemma D.3, we then have an embedding of Lie algebras,

$$\mathfrak{so}(Q) \subset C(Q)^{\mathrm{even}} \cong \mathfrak{gl}(\mathrm{Ext}^{\mathrm{even}} W) \oplus \mathfrak{gl}(\mathrm{Ext}^{\mathrm{odd}} W) \; , \tag{D.75}$$

and we find that there are two representations of $\mathfrak{so}(Q) = \mathfrak{so}_{2n}\mathbb{C}$. We denote the two representations by,

$$S^+ = \mathrm{Ext}^{\mathrm{even}} W \; , \quad S^- = \mathrm{Ext}^{\mathrm{odd}} W \; . \tag{D.76}$$

**Proposition D.4.** *The representations $S^{\pm}$ are the irreps of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weights $\alpha = \frac{1}{2}(L_1 + ... + L_n)$ and $\beta = \frac{1}{2}(L_1 + ... + L_{n-1} - L_n)$. More precisely, we have,*

$$S^+ = \Gamma_\alpha \; , \quad S^- = \Gamma_\beta \; , \quad \textit{if } n \textit{ is even} \; ; \tag{D.77}$$
$$S^+ = \Gamma_\beta \; , \quad S^- = \Gamma_\alpha \; , \quad \textit{if } n \textit{ is odd} \; . \tag{D.78}$$

*Proof.* We need to show that the natural basis vectors $e_I = e_{i_1} \wedge ... \wedge e_{i_k}$ for $\mathrm{Ext}^{\bullet} W$ are weight vectors. Tracing through the isomorphisms, we find that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+i})$ in $\mathrm{Ext}^2 V$, which corresponds to $\frac{1}{2}(e_i \cdot e_{n+i} - 1)$ in $C(Q)$, and this maps to,

$$\frac{1}{2}\left(L_{e_i} \circ D_{2e_i^*} - I\right) = L_{e_i} \circ D_{e_i^*} - \frac{1}{2} I \in \mathrm{End}(\mathrm{Ext}^{\bullet} W) \; . \tag{D.79}$$

We can compute,

$$L_{e_i} \circ D_{e_i^*}(e_I) = \begin{cases} e_I & \text{if } i \in I \\ o & \text{if } i \notin I \end{cases} \tag{D.80}$$

. So $e_I$ spans a weight space with weight $\frac{1}{2}\left(\sum_{i \in I} L_i - \sum_{j \notin I} L_j\right)$. All such weights with given $|I| \mod 2$ are congruent by the Weyl group (they are equivalent up to transformations of the Weyl group), so $S^+ = \mathrm{Ext}^{\mathrm{even}} W$ and $S^- = \mathrm{Ext}^{\mathrm{odd}} W$ must be an irreducible representation. The highest weights are then straightforward to read off - the one for $\mathrm{Ext}^{\mathrm{even}} W$ is $\alpha = \frac{1}{2}\sum_i L_i$ if $n$ is even and $\beta$ if $n$ is odd. $\square$

**Definition D.12.** The representations $S^{\pm}$ are the **half-spin representations** of $\mathfrak{so}_{2n}\mathbb{C}$, and $S = S^+ \oplus S^- = \mathrm{Ext}^{\bullet} W$ is called the **spin representation**. Elements of $S$ are known as **spinors**.

We are going to come back to spinor representations in the next section.

**Case 2:** $n = \dim V$ **is odd.**

This time we decompose the space $V$ as follows,

$$V = W \oplus W' \oplus U \ , \tag{D.81}$$

where $W$ and $W'$ are $n$-dimensional isotropic spaces and $U$ is a one-dimensional space perpendicular to them. For the standard $Q$ on $\mathbb{C}^{2n+1}$ these are spanned by the first $n$, second $n$, and the last basis vector. We then have the following lemma.

**Lemma D.5.** *The decomposition $V = W \oplus W' \oplus U$ determines an isomorphism of algebras,*

$$C(Q) \cong \mathrm{End}(\mathrm{Ext}^\bullet W) \oplus \mathrm{End}(\mathrm{Ext}^\bullet W') \ . \tag{D.82}$$

*Proof.* We can proceed exactly as the even case, as in Lemma D.4. The only difference is with the element $u_0$ where $Q(u_0, u_0) = 1$. We send $u_0$ to the endomorphism that is the identity on $\mathrm{Ext}^{\mathrm{even}} W$ and minus the identity on $\mathrm{Ext}^{\mathrm{odd}} W$. This involution then skew commutes with all $L_w$ and $D_\vartheta$, which means the map $V \to E = \mathrm{End}(\mathrm{Ext}^\bullet W)$ determines an algebra homomorphism from $C(Q) \to E$. The map for $\mathrm{End}(\mathrm{Ext}^\bullet W')$ is similar but with the roles of $W$ and $W'$ reversed. The maps are isomorphic by checking the basis elements. □

From Lemma D.5 we see that the subalgebra $C(Q)^{\mathrm{even}} \subset C(Q)$ ia mapped isomorphically onto the factors,

$$C(Q)^{\mathrm{even}} \cong \mathrm{End}(\mathrm{Ext}^\bullet W) \tag{D.83}$$

which gives a representation $S = \mathrm{Ext}^\bullet W$ of Lie algebras,

$$\mathfrak{so}_{2n+1}\mathbb{C} = \mathfrak{so}(Q) \subset C(Q)^{\mathrm{even}} \cong \mathfrak{gl}(\mathrm{Ext}^\bullet W) = \mathfrak{gl}(S) \ . \tag{D.84}$$

So now we have the following proposition.

**Proposition D.5.** *The representation $S = \mathrm{Ext}^\bullet W$ is the irrep of $\mathfrak{so}_{2n+1}\mathbb{C}$ with the highest weight*

$$\alpha = \frac{1}{2}(L_1 + ... + L_n) \ . \tag{D.85}$$

*Proof.* This is similar to the even case - each $e_I$ is an eigenvector with weight $\frac{1}{2}\left(\sum_{i\in I} L_i - \sum_{j\notin I} L_j\right)$. All the weights are congruent by the Weyl group so it must be an irrep with highest weight $\alpha$. □

We have therefore constructed the spin representations of $\mathfrak{so}_n\mathbb{C}$.

Let us summarise what we have done. We have constructed Clifford algebras via two methods — both abstractly and practically, and then looked at the structure of Clifford algebra itself. We found that the full Clifford algebra consists of the identity $\mathbb{1}$, the $D$

generating elements $\Gamma_M$, and all the independent matrices formed from products of the generators, given by the antisymmetric products,

$$\Gamma_{AB\ldots C} = \Gamma_{[AB\ldots C]} \; . \tag{D.86}$$

Under the canonical automorphism defined on the generators by $\Gamma_M \to -\Gamma_M$, the Clifford algebra inherits a natural $\mathbb{Z}_2$-grading, and it is under this that the Clifford algebra decomposes into even and odd subspaces which consists of real linear combinations of products of an even or odd number of gamma matrices respectively. In particular, the even subspace $C(Q)^{\text{even}}$ is a subalgebra, and we have seen how $\mathfrak{so}(p,q)$ is naturally embedded as a subalgebra in $C(Q)^{\text{even}}$.

## D.3 Spinor Representations and Clifford Algebras

Having discussed the spin representations of $\mathfrak{so}_n\mathbb{C}$, it is prudent to discuss its relation with the spinors in this subsection.

### D.3.1 Pin and Spin Groups

First, let us define something known as pin and spin groups.

**Definition D.13.** The **multiplicative group of units** in the Clifford algebra is defined to be the subset

$$C^\times(Q) = \left\{\phi \in C(Q) \,\middle|\, \exists \phi^{-1},\ \phi^{-1}\phi = \phi\phi^{-1} = 1\right\} \; . \tag{D.87}$$

This group contains all elements $v \in V$ with $Q(v) \neq 0$.

The group of units always acts naturally as automorphisms of the algebra, i.e. the **adjoint representation**,

$$\mathrm{Ad} : C^\times(Q) \to \mathrm{Aut}(C(Q)) \; , \tag{D.88}$$

which is given by,

$$\mathrm{Ad}_\phi(x) = \phi x \phi^{-1} \; . \tag{D.89}$$

Taking the derivation of this map gives the usual Lie bracket action $\mathrm{ad}_y(x) = [y, x]$. Hiterto we have assumed that the characteristic of the field could be any integer. Let us assume from now that the chracterisitc of the field $k \neq 2$. Then we have the following important proposition.

**Proposition D.6.** *Let $v \in V \subset C(Q)$ be an element with $Q(v) \neq 0$. Then $\mathrm{Ad}_v(V) = V$, and $\forall w \in V$, we have,*

$$-\mathrm{Ad}_v(w) = w - 2\frac{Q(v,w)}{Q(v)} v \; . \tag{D.90}$$

*Proof.* We have that $v^{-1} = -\frac{v}{Q(v)}$, so

$$-Q(v)\,\mathrm{Ad}_v(w) = -Q(v) v w v^{-1} = v w v = -v^2 w - 2Q(v,w) v = Q(v) w - 2Q(v,w) v \; . \tag{D.91}$$

□

Naturally, this lead us to consider the subgroup of elements $\phi \in C^\times(Q)$ such that $\mathrm{Ad}_\phi(V) = V$. From Proposition D.6 above, we see that the group contains all the elements $v \in V$ with $Q(v) \neq 0$, and when this happens the transformation $\mathrm{Ad}_v$ preserves the quadratic form $Q$,

$$(\mathrm{Ad}_v^* Q)(w) = Q(\mathrm{Ad}_v(w)) = Q(w) \ , \tag{D.92}$$

for all $w \in V$. We define $P(Q) \subset C(Q)$ to be the subgroup generated by the elements $v \in V$ with $Q(v) \neq 0$. Note that then there is a representation,

$$P(Q) \to O(V, Q) \ , \tag{D.93}$$

where

$$O(V, Q) = \{\lambda \in GL(V) \,|\, \lambda^* Q = Q\} \tag{D.94}$$

is the orthogonal group of the form $Q$.

We are now ready to explore the important subgroups of $P(Q)$.

**Definition D.14.** The **Pin group** is the subgroup of $P(Q)$ generated by the elements $v \in V$ with $Q(v) = \pm 1$, i.e.

$$\mathrm{Pin}(V) = \left\{ v_1 \cdot ... \cdot v_m \in C(Q) \,\middle|\, v_j \in S^{n-1} \subset \mathbb{F}^n, m \in \mathbb{N} \right\} \ . \tag{D.95}$$

**Definition D.15.** The Spin group is similarly defined as,

$$\mathrm{Spin}(V) = \mathrm{Pin}(n) \cap C^0(Q) \tag{D.96}$$

$$= \left\{ v_1 \cdot ... \cdot v_m \in C(Q) \,\middle|\, v_j \in S^{n-1} \subset \mathbb{F}^n, m \in 2\mathbb{N} \right\} \ . \tag{D.97}$$

Note that from the definition of the Pin group, the inverse element to $v_1 \cdot ... \cdot v_m$ is,

$$(v_1 \cdot ... \cdot v_m)^{-1} = (-v_m)...(-v_1) \in \mathrm{Pin}(V) \ . \tag{D.98}$$

Let us take a deep look at Eq. (D.90). Notice that the right hand side of the equation is basically the reflection $R_v(x)$ of the vector $x \in V$ where $v$ is the vector marking the perpendicular direction of the reflection hyperplane. To remove this sign we therefore consider the following action.

**Definition D.16.** The **twisted adjoint representation** is the map $\tilde{\mathrm{Ad}} : C^\times(Q) \to GL(C(Q))$ where,

$$\tilde{\mathrm{Ad}}_\phi(y) = \alpha(\phi) y \phi^{-1} \ . \tag{D.99}$$

For even elements $\phi$, $\tilde{\mathrm{Ad}}_\phi = \mathrm{Ad}_\phi$. We also have $\tilde{\mathrm{Ad}}_{\phi_1\phi_2} = \tilde{Ad}_{\phi_1} \circ \tilde{Ad}_{\phi_2}$. Explicitly,

$$\tilde{\mathrm{Ad}}_v(w) = w - 2\frac{Q(v, w)}{Q(v)} v \ . \tag{D.100}$$

We state without proof the following result.

**Theorem D.2** (Cartan-Dieudonné)**.** *Every $g \in O(V)$ is the product of a finite number of reflections $g = R_{u_1} \circ ... \circ R_{u_r}$ along the null lines where $Q(u_i) \neq 0$ and $r \leq \dim V$.*

*Proof.* See [22]. $\square$

We note that the twisted adjoint action must define a group homomorphism $\tilde{\text{Ad}} : \text{Pin}(V) \to O(V)$. It follows from the Cartan-Diedonné Theorem D.2 that $\tilde{\text{Ad}}$ is surjective. But what is the kernel of $\tilde{\text{Ad}}$?

**Proposition D.7.** *Suppose $V$ is finite dimensional and $Q$ is non-degenerate. Then the kernel of the homorphism $\tilde{\text{Ad}} : \tilde{P}(V,Q) \to GL(V)$ is the group $\mathbb{K}^\times$ of non-zero multiples of 1. Here the group $\tilde{P}(V,Q)$ is defined as,*

$$\tilde{P}(V,Q) = \left\{ \phi \in C^\times(V,Q) \,\middle|\, \tilde{\text{Ad}}_\phi(V) = V \right\} , \tag{D.101}$$

*where $P(V,Q) \subset \tilde{P}(V,Q)$.*

*Proof.* See [10] for a complete proof. The proof is also outlined in [11]. $\square$

[10] goes into a bit more detail in how you would define the homomorphism from the group $\tilde{P}(V,Q)$ to the orthogonal group $O(V)$ (see Propositions 2.5 and Corollary 2.6 of [10]). It also shows how the images $\tilde{\text{Ad}}(\text{Pin}(V,Q))$ and $\tilde{\text{Ad}}(\text{Spin}(V,Q))$ is a normal subgroup of $O(V)$ (see Proposition 2.8 of [10]). To summarise there are two exact short sequences.

**Theorem D.3.** *Let $V$ be a finite-dimensional vector space over a field $\mathbb{K}$ and $Q$ a non-degenerate quadratic form on $V$. Suppose the field $\mathbb{K}$ of characteristic $\neq 2$ is **spin**, i.e. at least one of the two equations $t^2 = a$ and $t^2 = -a$ can be solved in $\mathbb{K}$ for each non-zero element $a \in \mathbb{K}^\times$. Then there are two short exact sequences.*

$$0 \to F \to Spin(V,Q) \xrightarrow{\tilde{\text{Ad}}} SO(V) \to 1 , \tag{D.102}$$

$$0 \to F \to Pin(V,Q) \xrightarrow{\tilde{\text{Ad}}} O(V) \to 1 , \tag{D.103}$$

*where*

$$F = \begin{cases} \mathbb{Z}_2 = \{1, -1\} & \text{if } \sqrt{-1} \notin \mathbb{K} \\ \mathbb{Z}_4 = \{\pm 1, \pm\sqrt{-1}\} & \text{otherwise} \end{cases} \tag{D.104}$$

*The sequences above hold for general fields provided that $SO(V)$ and $O(V)$ are replaced by appropriate normal subgroups of $O(V)$ (since the map $\tilde{\text{Ad}}$ maps to normal subgroups of $O(V)$ in general and field $\mathbb{K}$, which is spin, solves the equation $t^2 Q(v) = \pm 1$ so every $v \in V^\times$ can be renormalised to have $Q(v) = 1$).*

*Proof.* See [10] Theorem 2.9. The details of the field being **spin** is not relevant if we restrict to $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ as both fields are spin. $\square$

The real case of the above Theorem D.3 is reduced to the simple short exact sequences,

$$0 \to \mathbb{Z}_2 \to \text{Spin}_{r,s} \to SO_{r,s} \to 1 , \tag{D.105}$$

$$0 \to \mathbb{Z}_2 \to \text{Spin}_{r,s} \to SO_{r,s} \to 1 , \tag{D.106}$$

for all $(r, s)$, where the subscripts denote the signature of the quadratic form $Q$. In particular, for $SO_n = SO_{n,0} = SO_{0,n}$, we have

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_n \xrightarrow{\tilde{\mathrm{Ad}}} SO_n \to 1 \ , \tag{D.107}$$

where the map $\tilde{\mathrm{Ad}}$ acts as the universal covering map of $SO_n$ for all $n \geq 3$. From this we see that the Pin and Spin groups are covering groups of the orthogonal group. We will now examine the representations of the Pin and Spin groups, which in term will give representations of the orthogonal groups. We will see how this unifies with the picture we have taken in two subsections ago.

### D.3.2 Classification of Clifford algebras

Let me say something about the classification of Clifford algebras. This turns out to be very useful in characterising the presence of spinor representations in different dimensions. From this point on we will switch notations from using $C(Q)$ to $\mathrm{Cl}(s, t)$, where the quadratic form $Q$ has signature $s$ and $t$. We will also define the complexified Clifford algebras as,

$$\mathbb{Cl}(s + t) = \mathrm{Cl}(s, t) \otimes_{\mathbb{R}} \mathbb{C} \ . \tag{D.108}$$

With this we can build up a chessboard of Clifford algebras. Firstly, the low-dimensional Clifford algebras can be explicitly checked.

**Lemma D.6.** *The low-dimensional Clifford algebras are given by,*

$$Cl(1, 0) \cong \mathbb{C} \ , \quad Cl(0, 1) \cong \mathbb{R} \oplus \mathbb{R} \ . \tag{D.109}$$

$$Cl(2, 0) \cong \mathbb{H} \ , \quad Cl(1, 1) \cong Mat_2(\mathbb{R}) \cong Cl(0, 2) \ . \tag{D.110}$$

*Proof.* Explicit computation. □

Then there are the periodicity conditions.

**Theorem D.4.** *There are isomorphisms (step isomorphisms),*

$$Cl(n, 0) \otimes Cl(0, 2) \cong Cl(0, n + 2) \ , \tag{D.111}$$

$$Cl(0, n) \otimes Cl(2, 0) \cong Cl(n + 2, 0) \ , \tag{D.112}$$

$$Cl(n, m) \otimes Cl(1, 1) \cong Cl(n + 1, m + 1) \ . \tag{D.113}$$

*Proof.* Proof of these are basically of the following format — take the gamma matrices $\Gamma'$ from $\mathrm{Cl}(n, 0)$ and $\Gamma''$ from $\mathrm{Cl}(0, 2)$ respectively and construct a new basis,

$$\Gamma_a = \begin{cases} \Gamma'_a \otimes \Gamma''_1 \Gamma''_2 & \text{for } 1 \leq a \leq d \ , \\ \mathbb{1} \otimes \Gamma''_{a-d} & \text{for } a = d + 1, \, d + 2 \ . \end{cases} \tag{D.114}$$

Then we can show that $\Gamma_a$ are the matrices for $\mathrm{Cl}(0, n + 2)$. The detailed proof is in [10, 18]. □

We will also need the elementary algebraic facts.

**Proposition D.8.** *The following hold true,*

1. $Mat_n(\mathbb{R}) \otimes Mat_m(\mathbb{R}) \cong Mat_{nm}(\mathbb{R})$.
2. $Mat_n(\mathbb{R}) \otimes_{\mathbb{K}} \mathbb{K} \cong Mat_n(\mathbb{K})$.
3. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.
4. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong Mat_2(\mathbb{C})$.
5. $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong Mat_4(\mathbb{R})$.

*Proof.* See Proposition 4.2 of [10]. □

**Theorem D.5.** *We have additional periodicity conditions.*

$$Cl(n+8,0) \cong Cl(n,0) \otimes Cl(8,0) \ , \tag{D.115}$$

$$Cl(0,n+8) \cong Cl(0,n) \otimes Cl(0,8) \ , \tag{D.116}$$

$$\mathbb{C}\mathrm{l}(n+2) \cong \mathbb{C}\mathrm{l}(n) \otimes_{\mathbb{C}} \mathbb{C}\mathrm{l}(2) \ , \tag{D.117}$$

*with*

$$Cl(0,8) = Cl(8,0) = \mathbb{R}(16) \ , \quad \mathbb{C}\mathrm{l}(2) = Mat_2(\mathbb{C}) \ . \tag{D.118}$$

The table obtained is as follows. This table can be further expanded by using the step

| $n$ | $\mathrm{Cl}(n,0)$ | $\mathrm{Cl}(0,n)$ | $\mathbb{C}\mathrm{l}(n)$ |
|---|---|---|---|
| 1 | $\mathbb{C}$ | $\mathbb{R} \oplus \mathbb{R}$ | $\mathbb{C} \oplus \mathbb{C}$ |
| 2 | $\mathbb{H}$ | $\mathrm{Mat}_2(\mathbb{R})$ | $\mathrm{Mat}_2(\mathbb{C})$ |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathrm{Mat}_2(\mathbb{C})$ | $\mathrm{Mat}_2(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C})$ |
| 4 | $\mathrm{Mat}_2(\mathbb{H})$ | $\mathrm{Mat}_2(\mathbb{H})$ | $\mathrm{Mat}_4(\mathbb{C})$ |
| 5 | $\mathrm{Mat}_4(\mathbb{C})$ | $\mathrm{Mat}_2(\mathbb{H}) \oplus \mathrm{Mat}_2(\mathbb{H})$ | $\mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_4(\mathbb{C})$ |
| 6 | $\mathrm{Mat}_8(\mathbb{R})$ | $\mathrm{Mat}_4(\mathbb{H})$ | $\mathrm{Mat}_8(\mathbb{C})$ |
| 7 | $\mathrm{Mat}_8(\mathbb{R}) \oplus \mathrm{Mat}_8(\mathbb{R})$ | $\mathrm{Mat}_8(\mathbb{C})$ | $\mathrm{Mat}_8(\mathbb{C}) \oplus \mathrm{Mat}_8(\mathbb{C})$ |
| 8 | $\mathrm{Mat}_{16}(\mathbb{R})$ | $\mathrm{Mat}_{16}(\mathbb{R})$ | $\mathrm{Mat}_{16}(\mathbb{C})$ |

**Table D.1**: Table for classification of Clifford algebras.

isomorphisms into different $\mathrm{Cl}(s,t)$. I won't do it here, but you should have a look at [10] for the nice table.

### D.3.3 Spinor Representations

Before we begin let us recall what representations of an algebra is.

**Definition D.17.** Suppose $A$ is an associative algebra and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. A **$\mathbb{K}$-representation** of $A$ is a $\mathbb{K}$-linear homomorphism $\rho : A \to \mathrm{End}_{\mathbb{K}}(E)$ for some $\mathbb{K}$-vector space $E$. Two $\mathbb{K}$-representations $\rho : A \to \mathrm{End}_{\mathbb{K}}(E)$ and $\rho' : A \to \mathrm{End}_{\mathbb{K}}(E')$ are **equivalent** if there exists a $\mathbb{K}$-linear isomorphism $f : E \to E'$ such that the following triangle commutes:

$$\begin{tikzcd} & A \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] & \\ \mathrm{End}_{\mathbb{K}}(E) \arrow[rr, "\mathrm{Ad}\, f"] & & \mathrm{End}_{\mathbb{K}}(E') \end{tikzcd}$$

where $\mathrm{Ad}\, f : \mathrm{End}_{\mathbb{K}}(E) \to \mathrm{End}_{\mathbb{K}}(E')$ is defined as $\phi \mapsto f \circ \phi \circ f^{-1}$, so $f \circ \rho(a) = \rho'(a) \circ f$ for all $a \in A$.

We can now define the following representations.

**Definition D.18.** A **pinor representation** of $\mathrm{Pin}(V)$ is the restriction of an irreducible representation of $C(Q)$. Similarly, a **spinor representation** of $\mathrm{Spin}(V)$ is the restriction of an irreducible representation of $C_0(Q)$.

It is actually good to stop and define what representations we are exactly shooting for. In physics we encounter four types of spinors — Weyl, Dirac, Majorana and Weyl-Majorana spinors. Let us first give a proper definition, and then we will see how our classification of Clifford algebras above actually give us the full classification of spinor representations in each signature.

**Definition D.19.** Suppose $\mathrm{Cl}(n)$ is the Clifford algebra of $\mathbb{R}^n$ with its complexification given by $\mathbb{Cl}(n) = \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathbb{C}$. For the standard basis $e_i$ of $\mathbb{R}^n$, define

$$z_j = \frac{1}{2}(e_{2j-1} - ie_{2j}) \in \mathbb{Cl}_n \tag{D.119}$$

for $j = 1, ..., m$, $n = 2m$ and $\bar{z}_j$ be its conjugate. Then the span

$$\Sigma = \{z_{j_1} \cdot ... z_{j_k} \cdot \bar{z}_1 \cdot ... \cdot \bar{z}_m \,|\, k = 0, ..., m, \quad 1 \leq j_1 \leq ... \leq j_k \leq m\} \ , \tag{D.120}$$

defines a complex vector subspace of $\mathbb{Cl}_n$ of dimension $2^m$. This space that is known as the **spinor space** with its elements being **spinors**. The spinor space is invariant under Clifford multiplications (multiplication by $e_j$). We further define $\Sigma^{\pm}$ to be the spans where $k$ is even and odd respectively, and $k$ is known as the **chirality** of the spinor space.

We see how the spinor representation is exactly $E = \Sigma$ in the Definition D.17. Note very carefully that the dimensions of the spinor space $\Sigma$ has nothing to do with the dimension of $V$ which is the vector space where we have defined the Clifford algebra. This is important for the classification of spinors. Most representations of Clifford algebras (similar to Lie algebras) are reducible. The volume element plays a key role in determining irreducible representations. In particular, since the volume element is used in determining the classification of real and complex Clifford algebras [10], one can use it to determine the properties of irreducible real, complex and quaternionic representations of Clifford algebra. You can read about the details of the classification and determination of irreducible representations in [10] and [11] — I am simply going to make some summary statements here.

We start with complex Clifford algebras (so we can take complex linear combinations of products of gamma matrices). The reason for doing that is that the structure of complexified algebras typically gives uniformity in their representation theory (think Lie algebras [19]!). Then we characterise their representations as follows.

**Definition D.20.** A **Dirac spinor** is the fundamental complex spinor representation of $\mathrm{Cl}_0(s,t)$, or the map $\Delta_n^{\mathbb{C}} : \mathrm{Spin}(n) \to GL_{\mathbb{C}}(\Sigma)$. This representation is irreducible for odd $n$.

In even dimensions something specific happens and we have the following kind of spinors.

**Definition D.21.** A **Weyl spinor** is a complex, irreducible representation of $\mathrm{Cl}_0(s,t)$ in even dimensions.

Okay. How do we understand this categorisation? Recall that to understand spinor representations, the chain of embedding $\mathrm{Spin}(s,t) \subset \mathrm{Cl}_0(s,t) \subset \mathbb{C}\mathrm{l}_0(s+t)$ requires us to look at even subalgebras. There is a fundamental isomorphism given by the following.

**Proposition D.9.** *There is an algebra isomorphism between even subalgebras of Clifford algebra and one of a higher dimension.*

$$Cl(s,t) \cong Cl_0(s+1,t) \ , \quad \text{for } s \geq 1 \ , \tag{D.121}$$

$$Cl(n) \cong Cl_0(n+1) \ . \tag{D.122}$$

*Complexification therefore gives,*

$$\mathbb{C}\mathrm{l}(n) \cong \mathbb{C}\mathrm{l}_0(n+1) \ . \tag{D.123}$$

*Proof.* See Theorem 3.7 of [10]. □

This gives the following categorisation in Table D.2. From this table, we see that in

| $d \mod 2$ | $\mathbb{C}\mathrm{l}_0(d)$ | $N$ |
|---|---|---|
| 0 | $\mathrm{Mat}_N(\mathbb{C}) \oplus \mathrm{Mat}_N(\mathbb{C})$ | $2^{(d-2)/2}$ |
| 1 | $\mathrm{Mat}_N(\mathbb{C})$ | $2^{(d-1)/2}$ |

**Table D.2**: Even subalgebras of a complex Clifford algebra. $N$ is the dimension of the algebra, not the underlying vector space $d$.

even dimensions $d$ there are two spinor representations of dimension $2^{(d-2)/2}$. These are distinguished by the volume element,

$$\Gamma_{d+1} = \alpha \Gamma_1 \Gamma_2 \ldots \Gamma_d \ , \tag{D.124}$$

with $\alpha \in \mathbb{C}$ such that $\Gamma_{d+1}^2 = \mathbb{1}$. Typically $\alpha = (-i)^{\frac{d}{2}+1}$ is picked [9]. This $\Gamma_{d+1}$, sometimes denoted as $\Gamma_*$, is the highest rank Clifford algebra element, and it (anti-)commutes with all (odd) even rank elements of the algebra,

$$[\Gamma_{d+1}, \mathrm{Cl}_0(d)] = 0 \ . \tag{D.125}$$

This is sometimes known as the **chirality operator**, also defined as the image of the volume element in the Clifford algebra in the representation. From this we can define the projection operators in even dimensions,

$$P_{L,R} = \frac{1}{2}(\mathbb{1} \pm \Gamma_{d+1}) \ , \tag{D.126}$$

and hence project the representation space (or a Clifford module $\Delta_n^{\mathbb{C}}$) to the two eigenspaces of $\Gamma_{d+1}$, with eigenvalues $\pm$. This distinguishes the two irreducible representations as *Weyl spinors*. Our argument can be summarised in the following proposition.

**Proposition D.10.** *Suppose the complex spinor representation is $\Delta_n^{\mathbb{C}} : Spin_n \to GL_{\mathbb{C}}(\Sigma)$ which is given by restricting an irreducible complex representation $\mathbb{Cl}_n \to \mathrm{Hom}_{\mathbb{C}}(\Sigma, \Sigma)$ to $Spin_n \subset Cl_n^0 \subset \mathbb{Cl}_n$. Then for $n$ add this definition is independent of which irrep of $\mathbb{Cl}_n$ is used, and that the representation $\rho_n^{\mathbb{C}}$ is irreducible. When $n$ is even then there is a decomposition,*

$$\Delta_n^{\mathbb{C}} = \Delta_n^{\mathbb{C}^+} \oplus \Delta_n^{\mathbb{C}^-} \ , \tag{D.127}$$

*i.e. into a direct sum of two inequivalent irreducible complex representations of $Spin_n$.*

*Proof.* See Proposition 5.15 of [10], which uses the argument for the real case. For more information one can also look at [23]. □

Let us move on to real representations. There are two approaches here — the traditional one starts with the complexified form of the Clifford algebras and look at their representations that comes with a real structure. In this sense, the definition of the spinors is as follows.

**Definition D.22.** Let $S$ be a spinor representation of a complexified Clifford algebra. We say that $S$ is **Majorana** if $S$ admits a real structure $\mathcal{J}$. A spinor $\psi \in S$ is **Majorana** if it satisfies $\mathcal{J}(\psi) = \psi$.

**Definition D.23.** Let $S \times V$ be a spinor with flavour representation of a complexified Clifford algebra and some group (with $V$ some quaternionic representation of $G$), $\mathrm{Spin}(s,t) \times G$. We say that $S \times V$ is **symplectic Majorana** if $S \times V$ admits a quaternionic structure $\mathcal{J}_\otimes$. A spinor $\psi \in S$ is **symplectic Majorana** if it satisfies $\mathcal{J}_\otimes(\psi) = \psi$.

**Definition D.24.** In the even dimensions where a real structure $\mathcal{J}$ and the chirality projection operator $\Gamma_{d+1}$ both exists, a spinor is **Majorana-Weyl** if

$$\mathcal{J}(\psi) = \psi \ , \quad \Gamma_{d+1}\psi = \pm i\psi \ . \tag{D.128}$$

Similarly, if $\mathcal{J}$ is a quaternion structure then the spinor that satisfies Eq. (D.128) is called **Majorana-symplectic-Weyl**.

Let me explain how we should interpret this. Firstly, we know that a complex representation that admits a real or quaternionic structure is equivalent to an invariant non-degenerate complex bilinear form $B$ satisfying,

$$B(\Gamma_a \cdot \psi_1, \psi_2) = \tau B(\psi_1, \Gamma_a \cdot \psi_2) \ . \tag{D.129}$$

Here $\tau$ is a sign and we also have

$$B(\psi_1, \psi_2) = \epsilon B(\psi_2, \psi_1) \ , \tag{D.130}$$

with $\epsilon = \pm 1$ depending on whether we encode a real or quaternionic structure. We can choose a matrix that represents $B$ by (apologies for using the same notation),

$$\Gamma_a^* = \tau B \cdot \Gamma_a \cdot B^{-1} \ . \tag{D.131}$$

In the classification in [9], the $\tau = -t_0 t_1$, and with

$$B^* B = -t_1 \mathbb{1} \ , \tag{D.132}$$

we can identify $\epsilon = -t_1$. This complex bilinear form gives the action of the real (quaternionic) structure $\mathcal{J} : S \to S$ via the charge conjugate map:

$$\mathcal{J} : \psi \mapsto B^{-1} \psi^* \ . \tag{D.133}$$

Now there is also a *charge conjugation matrix* $C$ which is related to $B$ in [9] via $B = i t_0 C \Gamma_0$. Alternatively, given $A$ the volume form (related to the non-degenerate Hermitian form),

$$\Gamma_a = -(-1)^t A \cdot \Gamma_a \cdot A^{-1} \ , \tag{D.134}$$

then,

$$C = t_0 B^T \cdot A \ . \tag{D.135}$$

The reason why $C$ is useful is because we have,

$$\Gamma_a^T = t_0 t_1 C \cdot \Gamma_0 \cdot C^{-1} \ , \quad C^\dagger \cdot C = 1 \ , \tag{D.136}$$

$$C^T = -t_0 C \ . \tag{D.137}$$

What does this imply? we see then $C\Gamma^{(r)}$ is either symmetric or antisymmetric,

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)} \ , \tag{D.138}$$

and we can now look at the possibility of different combinations of $t_0$ and $t_1$ that occurs in various dimensions. It turns out the following statements are true:

1. $t_1 = -1$ must hold true for the reality condition to hold, so investigate the cases. $t_0 = +1$ is only possible when $d \mod 8 = 2, 3, 4$. These are the **Majorana spinors**. For $t_0 = -1$ which works for $d \mod 8 = 4, 5, 6$ the spinors are known as **pseudo-Majorana spinors**. There are no real representations, the Clifford algebra generating gamma-matrices instead imaginary.

| $s-t \mod 8$ | $\mathrm{Cl}_0(s,t)$ | $N$ |
|---|---|---|
| $1,7$ | $\mathrm{Mat}_N(\mathbb{R})$ | $2^{(d-1)/2}$ |
| $3,5$ | $\mathrm{Mat}_N(\mathbb{H})$ | $2^{(d-3)/2}$ |
| $2,6$ | $\mathrm{Mat}_N(\mathbb{C})$ | $2^{(d-2)/2}$ |
| $4$ | $\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$ | $2^{(d-4)/2}$ |
| $0$ | $\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$ | $2^{(d-2)/2}$ |

**Table D.3**: Structure of the even subalgebras of a real Clifford algebra.

2. For $t_1 = 1$ we cannot define Majorana spinors, but must instead define **symplectic Majorana spinors** which satisfy the condition,

$$\chi^i = \epsilon^{ij} B^{-1} (\chi^j)^* , \quad \epsilon^{ij} = -\epsilon^{ji} . \tag{D.139}$$

   These are exactly the spinors with quaternionic structure above, the weird $\epsilon^{ij}$ encodes the quaternionic representation $V$.

3. Finally in $d \mod 8 = 2$, 4, we see that the Weyl and Majorana conditions are compatible so we have **Majorana-Weyl spinors**. For $d = 0 \mod 4$ dimensions we have,

$$(P_L \psi)^C = P_R \psi , \tag{D.140}$$

   so the Majorana condition cannot be satisfied.

A lot of this discussion can be simplified if we instead discuss directly the representation content of the real Clifford algebras. There we have Table D.3, From this table you can immediately read out the availabilities of spinor representations.

- For odd dimensions there is a unique spinor representation for $s - t = 1, 7 \mod 8$ — they are **Majorana spinors**.
- For odd dimensions there is a unique spinor representation for $s - t = 3, 5 \mod 8$ — they are **symplectic Majorana spinors**.
- For even dimensions we have two inequivalent representations.
- For $s-t = 2, 6 \mod 8$ there are two inequivalent complex representations (remember the table is before complexification), labelled by chirality — they are **Weyl spinors**.
- For $s - t = 0 \mod 8$ there are two inequivalent complex representations compatible with the real structure — they are **Majorana-Weyl spinors**.
- For $s - t = 4 \mod 8$ there are two inequivalent quaternionic spinor representations — they are **symplectic Majorana-Weyl spinors**.

Note that in the case where $s - t \mod 8 = 2$, i.e. in the case where we have 4d Lorentzian signature, the matrix algebra is $\mathrm{Mat}_N(\mathbb{C})$. The table only gives us the structure of the even subalgebra of the real Clifford algebra, so in this case we know that there is a natural complex module. This however does not rule out a real module (i.e. a Majorana spinor)! Majorana spinors are, at the end of the day, real representations of the Clifford algebra, so we will need to characterise the spinor representations by the real structure.

There is a lot more with the technology of Clifford algebras. You can read [10, 11, 18, 24–26] when you have the time, but I digressed.

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