Prepared for submission to JHEP

Supersymmetry Example Classes 2026

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Abstract: A set of class notes for the Supersymmetry and Supergravity course at Oxford in Hilary Term 2026.

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0 Introduction

These set of notes are for the Supersymmetry and Supergravity classes in 2026.

Supersymmetry, in my opinion, is still a very important topic to study. It is still used in many BSM phenomenology research, mathematical research and of course, string theory. Below is my very poor attempt to complement Michèle's amazing course which is based mostly on [1]. I wanted to give a more complete picture in the classes, so below is my rudimentary attempt to do that.

This set of notes is riddled with mistakes, so please let me know if you find any. A quick health warning too — most of the appendices are really just my own notes (because I like maths and physics) so if you don't like formal stuff, don't read anything beyond Appendix A. I don't want to be responsible for anyone getting hurt.

1 Spurions, Naturalness, the Hierarchy Problem and SUSY

Supersymmetry is the symmetry between fermions and bosons. To put it simply, there is a natural splitting of the Hilbert space $\mathcal{H}$ as,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F \ , \tag{1.1}$$

where $\mathcal{H}_B$ and $\mathcal{H}_F$ indicates the Hilbert space with an even and odd number of fermionic excitations respectively. The operator $\mathcal{Q}$,

$$\mathcal{Q} : \mathcal{H}_{B,F} \to \mathcal{H}_{F,B} \tag{1.2}$$

with the following two properties:

$$\mathcal{Q}^2 = 0 \ , \tag{1.3}$$

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2H \ . \tag{1.4}$$

Here H is the Hamiltonian of the theory. There are immediately two consequences of having this symmetry generated by $\mathcal{Q}$.

1. $[H, \mathcal{Q}] = 0$. This $\mathcal{Q}$ actually commutes with the Hamiltonian so it is a symmetry.
2. $\langle\Psi| H |\Psi\rangle \geq 0$ for any $|\psi\rangle \in \mathcal{H}$, which is an equality if and only if $\mathcal{Q}|\psi\rangle = 0 = \mathcal{Q}^\dagger|\psi\rangle$. This means that for a supersymmetric vacua $E_0 = 0$ [1].

Before we continue, I want to begin by asking the question — why should we study supersymmetry? To do this, let me first introduce the concept of spurions.

1.1 Spurions and technical naturalness

Let us define spurions as follows.

Definition 1.1. A **spurion field** in a theory is a parameter which breaks an enhanced global symmetry $\mathcal{G}$.

This definition requires a bit of an explanation. Typically we can enhance the symmetry of a theory by allowing fictitious background fields to transform under a symmetry group. Such background fields can be constructed by effectively treating a parameter as a field — when this field takes a vacuum expectation value, this spuriously enhanced symmetry will then be spontaneously broken by the VEV (which is the parameter value). This field, is then known as a **spurion field**.

Let me illustrate this with an example. Consider a single complex scalar field and a Weyl fermion with the following Lagrangian,

$$\mathcal{L} = -\int d^4x \left[\partial_\mu\phi\partial^\mu\phi^* + i\psi^\dagger\bar{\partial}\!\!\!/\psi - M_\phi^2|\phi|^2 + \frac{1}{2}M_\psi\psi\psi\right] \ . \tag{1.5}$$

First note the global symmetries in the system when the mass terms are ignored. There are two global $U(1)$-symmetries, namely $\phi \mapsto e^{-i\theta'}\phi$ and $\psi \mapsto e^{-i\theta}\psi$; and a shift symmetry on the scalars, $\phi \mapsto \phi + c$ where c is a constant. Here we can distinguish the spurions associated to the global symmetries. Clearly, if we set $M_\phi \to 0$ then we recover the shift symmetry; whilst if we set $M_\psi \to 0$ we will get the chiral $U(1)$-symmetry recovered — hence M_ϕ and M_ψ are the spurions for the global shift and chiral symmetries respectively.

[1] This point, as we will see, will become important in SUSY-breaking.

Why is the idea of spurions important? Turns out spurions allow us to make order-of-magnitude estimates to parameter scales in our theory. Recall that in QFT we have learnt that any field theory could be (and arguably, should be) interpreted as an effective field theory with **scale cut-off** Λ. This scale cut-off gives the scale when theory breaks down and the description of the theory is no longer valid for the energies beyond that scale. If an EFT comes from a field theory of a higher energy cut-off (like in HEP), one can now ask the question — are the parameters of the low-energy effective theory *physically natural* — i.e. do we have to choose particular parameters in my high energy theory to make certain parameters in my low-energy theory? Dirac originally considered this and came up with a very simple approach [2].

Definition 1.2 (Dirac's naturalness principle.)**.** Suppose we have two QFTs, S_1 and S_2, with cut-offs Λ_1 and Λ_2 and $\Lambda_1 > \Lambda_2$. Then all the parameters in the low-energy theory must be at least of the order,

$$c^{(2)} \sim \mathcal{O}\left(\frac{\Lambda_2}{\Lambda_1}\right) . \tag{1.6}$$

In particular, $c^{(2)} \ll \mathcal{O}\left(\frac{\Lambda_2}{\Lambda_1}\right)$ will be unnatural.

This turns out to be too stringent of a requirement. The modern understanding of naturalness comes from 't Hooft [3] who introduced the idea of technical naturalness [2].

Definition 1.4 ('t Hooft naturalness)**.** The effective interactions of a theory at a scale $\Lambda_2 < \Lambda_1$ should follow from the properties of a theory at a scale Λ_1, without the requirement that various parameters in the high energy theory should match at the order $\sim \Lambda_2/\Lambda_1$. In particular, if a parameter $\alpha(\Lambda_2)$ is small in the low-energy theory, setting $\alpha(\Lambda_2) = 0$ must increase the symmetry of the system.

't Hooft's idea comes from the observation that parameters can be set small if it is a near symmetry — i.e. there is some approximate symmetries that we can break weakly to obtain this small parameter. Otherwise, the small parameter must come from some careful construction of small parameters in the high-energy scale or parameters of similar scale but differ by the order $\sim \mathcal{O}(\Lambda_2/\Lambda_1)$.

Translating into spurion language, this means the spurion can be small as it is parametrises the breaking of a global symmetry. Nice. Let us now look at the consequence of naturalness

[2] Note that this definition is different from the modern definition of *technical naturalness*.

Definition 1.3 (Technical naturalness)**.** Parameters in a theory are called **technically natural** if their size in the UV theory is not spoiled when renormalised down to the IR at intermediate scales.

This definition is important as it is possible for parameters to be technically natural but not 't Hooft natural — for these cases an assumption is made about the absence of additional physical thresholds (corrections due to integrating out high-energy modes) intermediate between the UV and the IR which could induce large corrections in the absence of a symmetry. An example would be the super-Higgs model — supersymmetry protects the renormalisation of the Higgs mass parameter m_H^2, but setting $m_H^2 \to 0$ does not recover any global symmetry. The physical thresholds are absence owing to the presence of supersymmetry.

in the quantum corrections of the spurion. Quantum loop order effects should correct the spurion field, and in general we will have, for c a spurion,

$$c_{\text{phys}} = c_0 + \delta c \ , \tag{1.7}$$

where c_0 is the parameter (the free field) in the theory and δc is the quantum corrections caused by loop effects in the theory. Let us think about what happens when we set $c \to 0$ to restore some global symmetry $\mathcal{G}$. Clearly, when we have a near symmetry, $c_{\text{phys}} \ll 1$. The bare coupling clearly contains divergences sensitive to the cut-off scale Λ and renormalisation scheme dependences that is cancelled by δc, so we in general have

$$\delta c = f(c, \Lambda) \ . \tag{1.8}$$

At this point $f(c, \Lambda)$ can be any function we want. But we can do better. Remember what a spurion field is. A trick in spurion analysis is to promote the spurion c to a field in a theory, $\tilde{c}$ which takes the parameter value as its vacuum expectation value,

$$\langle \tilde{c} \rangle = c \ . \tag{1.9}$$

What does this have to do with quantum corrections? Since setting $c \to 0$ restores the global symmetry, we must have $\delta c \to 0$ in this case. To put it more explicitly, suppose we now promote c to a field $\tilde{c}$ and compute the quantum corrections to its vacuum expectation value. Then since the global (spurion) symmetry is preserved, we must have,

$$\delta \langle \tilde{c} \rangle \sim \tilde{c} \ , \tag{1.10}$$

as both sides must have the same charge. In the simple model with a Weyl fermion above for example, we must have,

$$\delta M_\psi \sim M_\psi f(\Lambda, \dots) \tag{1.11}$$

where $f(\Lambda, \dots)$ must be a charge-invariant function. Why can't I add in some term which has the same charge as the spurion field M_ψ? Suppose we have at some energy Λ_M where $M_\psi = 0$. Quantum corrections will now mean that at lower energies M_ψ is regenerated as an operator — but $M_\psi \psi \psi$ is protected by the (spurious) chiral symmetry [3]! Therefore, $f(c, \Lambda)$ actually has the form,

$$f(c, \Lambda) = c\,(1 + \cdots) \ , \tag{1.12}$$

and have therefore arrived in the important result:

Proposition 1.1. *If a spurion in a theory breaks a particular symmetry, then the size of that parameter will not receive any large corrections in perturbation theory, so it is technically natural for it to be small.*

Note that here we have distinguished between a mass M_1 of a theory and a scale Λ_1. Very briefly said, the mass M_1 is where *new physics comes in*, whereas the scale Λ_1 tells you the highest energy that the theory has predictive power. So they are very different things [4].

[3] This means in the eyes of Wilsonian EFT, any field A with the same charge as M_ψ will generate the coupling $A\psi\psi$ which is the same as the mass term we start with. If we wish to restore the global chiral symmetry without any spuriousness, we must set $M_\psi = A = \cdots = 0$, and this term must not be regenerated under renormalisation.

[4] The interested among you can pursue this point of view further in [4]. My discussion of spurions is loosely based of his notes as well.

1.2 The renormalised operators and the naturalness problem

Let us go back to QFT and look at operators in an effective field theory. Recall that a general effective Lagrangian is defined by the field content and the symmetries. An operator of dimension d will have the coefficient in the Lagrangian,

$$\delta\mathcal{L}_{\text{eff}} \sim \frac{1}{\Lambda^{d-D}}\mathcal{O}_d \tag{1.13}$$

where D is the dimension of the theory. To calculate a physical quantity, we could imagine an expansion of the form,

$$\mathcal{A} \sim \mathcal{A}_0 \left[1 + \frac{M^{d-D}}{\Lambda^{d-D}} + \dots\right] , \tag{1.14}$$

with M the kinematic scale of the physical process and Λ the scale of the physics. We see how operators of dimension $d > D$ have less significant effects at low-energies — so we call them **irrelevant operators**. In contrast, the effects of the operators with dimension $d < D$ increase with energy so we call them **relevant**, whilst the ones with dimension $d = D$ are called **marginal** as they are independent of energy.

Let us now consider what happens to our parameters when we have relevant operators that cannot be forbidden by symmetries. Recall a model that all of you have seen in QFT (see Problem Sheet 4). We have a field theory consisting of a scalar with mass μ and a Dirac fermion ψ of mass m described by the Lagrangian density [5],

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2\right) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - g\phi\bar{\psi}\psi + \mathcal{L}_{\text{int}}(\phi) + \mathcal{L}_{\text{c.t.}} . \tag{1.15}$$

First let's apply spurion analysis. Notice that because of the interaction term (say $\mathcal{L}_{\text{int}}(\phi) = -\frac{\lambda}{4!}\phi^4$), there is now no global symmetry that is recovered when $\mu^2 \to 0$. We must simultaneously set $\lambda, \mu^2 \to 0$ for the shift symmetry $\phi \mapsto \phi + c$ to be restored. Therefore $-\frac{1}{2}\mu^2\phi^2$ is a renormalisable operator that are not forbidden by any global symmetry.

But consider now what happens when we carry out a one-loop calculation on μ^2. The renormalised scalar mass μ_R^2 can be calculated by using some regularisation scheme to obtain (schematically),

$$\mu_R^2(m) = \mu^2(m) + \frac{c_3 y^2}{16\pi^2}m^2 , \tag{1.16}$$

where c_3 is a constant that depends on the regularisation scheme and we have matched the scales at $\mu = m$. Notice the dependence on the fermion mass m. This means that to make the scalar light compared to the mass-scale m, we will need to tune the renormalised couplings in the fundamental theory such that the bare mass μ^2 cancels out with the term involving m^2 — there is no obvious symmetry principle motivating this, and we say that the scalar mass μ is fine-tuned.

This suggests the following statement.

[5] I would like to thank my office mate Gaurang for enlightening me this example.

Definition 1.5. The **naturalness problem for scalar mass parameters** states that the renormalised mass for scalar particles in a theory is quadratically dependent on some UV scale Λ_{UV}. The relevant operators that are not forbidden by symmetries are generally sensitive to heavy physical thresholds in the theory. This problem is regulator-independent — this dependence on UV scale will exist regardless of the regularisation scheme chosen.

To summarise, the naturalness problem arises as in general renormalisable couplings in our theory will have scale-dependent loop contributions to scalar mass parameters. This means that although the low-energy theory alone does not suffer from any problems [6], if we expect the fundamental theory to be finite and fully predictive, then these divergences mean by restricting ourselves in a subset of the full theory (the low-energy theory), we have neglected finite, physical contributions from the missing parts of the theory (in this case, the high-energy modes). This dependence on cut-off scales Λ, entirely removable by renormalisation, provides a useful proxy for the dependence on physical scales (UV-scales, Λ_{UV}) in a more UV-complete theory.

Let us go back to the theory defined in Eq.(1.15).

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2\right) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - g\phi\bar{\psi}\psi + \mathcal{L}_{\mathrm{int}}(\phi) + \mathcal{L}_{\mathrm{c.t.}} \ . \tag{1.15}$$

Remember in that question we were asked to calculate explicitly the counter-term for a scalar three-point function,

$$\delta\mathcal{L}_{\mathrm{c.t.}} \ni -\frac{1}{3!}\delta_\eta\phi^3 \ . \tag{1.17}$$

Why did we need this counter-term in the first place? Notice the presence of the Yukawa term in Eq.(1.15) breaks the $\mathbb{Z}_2$-symmetry,

$$\mathbb{Z}_2 : \phi \mapsto -\phi \ , \tag{1.18}$$

and g is a $\mathbb{Z}_2$-spurion. Therefore, loop corrections of parameters that breaks this symmetry must be accompanied with odd powers of g. In particular, after a long calculation, we will obtain,

$$\delta_\eta = -g^3 m \frac{3}{4\pi^2}\log\frac{\Lambda^2}{m^2} + f_\eta + \mathcal{O}(g^4) \tag{1.19}$$

Here, g is a spurion and it is technically natural. However, because of the $\mathbb{Z}_2$-symmetry, radiatively all terms that are invariant under the spurious symmetry (by promoting g to a field) $g \mapsto -g$ will be generated, in particular, the term,

$$\delta\mathcal{L} \sim \eta\phi^3 \tag{1.20}$$

will be radiatively generated with the correction given in Eq.(1.19). This now illustrates the **totalitarian principle** clearly — where anything that is not forbidden by a symmetry must be included (in this case, a spurious symmetry).

[6] By this I mean we can choose an appropriate renormalisation procedure to absorb the divergences encountered in loop calculations. You can also check that here μ^2 is not technically natural in the sense defined in Definition 1.3, since there are now threshold corrections ($\sim m^2$) when we do renormalisation.

1.3 The hierarchy problem

We can now state the hierarchy problem.

Definition 1.6 (The EW hierarchy problem)**.** The **electroweak hierarchy problem** is the naturalness problem that the Higgs mass being sensitive to the UV scale.

Let us illustrate this problem in a bit more detail [5]. The Standard Model can be treated as an effective field theory up to some cut-off scale $\Lambda_{\rm SM}$. We can compute the one-loop corrections to the Higgs mass,

$$\delta m_H^2 = \frac{\Lambda_{\rm SM}^2}{16\pi^2}\left(-6y_t^2 + \frac{9}{4}g^2 + \frac{3}{4}g'^2 + 6\lambda\right) . \tag{1.21}$$

If we treat the SM as an EFT with a cut-off $\Lambda_{\rm SM}$, recall from the previous subsection that we can simply interpret these divergences as corrections that can be cancelled out by an appropriate renormalisation scheme. However, we know from our physics education that the SM is not the end of the story. Gravity is not incorporated, and when we put the SM in some more complete theory with physical scales these divergences will be replaced by finite contributions dependent on $\Lambda_{\rm UV}$. In particular, the typical argument goes — when we set this $\Lambda_{\rm UV} \sim M_P$ where we expect quantum gravity effects to come in, then,

$$\delta m_H^2 \sim M_P^2 \gg m_H^2 , \tag{1.22}$$

where m_H^2 is the double mass parameter inferred from the Higgs VEV v and Higgs mass m_h^2. This is the known as the electroweak hierarchy problem, and it is by comparing scales that we know $\Lambda \sim 500$ GeV is when new physics should start to enter.

We however still don't know what the new physics is. And it is in this context that supersymmetry first came up as a plausible solution to new physics.

1.4 SUSY and hierarchy problem

Let us look at how SUSY saves the hierarchy problem — but we will first need to do some calculations. Recall that in the Standard Model the Higgs field H is a complex scalar with a scalar potential of form,

$$V = m_H^2|H|^2 + \lambda|H|^4 , \tag{1.23}$$

which gives the vacuum expectation value of

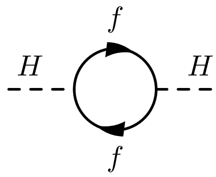
$$\langle H\rangle = \sqrt{-\frac{m_H^2}{2\lambda}} . \tag{1.24}$$

The problem however is that m_H^2 receives huge quantum corrections from virtual effects of all particle phenomenology that couples directly or indirectly to H, namely the term

$$\mathcal{L} \supset -\lambda_f H\bar{f}f . \tag{1.25}$$

Let us in particular calculate the one-loop contribution to m_H^2. The one-loop fermionic contribution is given by the following diagram,

$$\delta m_H^2 \quad = \quad$$



$$+ \ \ldots \qquad (1.26)$$

which is given by (using $f = t$ as the top quark gives the leading-order contributions)

$$-i\delta m_H^2|_{\text{top}} = (-1)N_c \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left[\frac{-iy_t}{\sqrt{2}} \frac{i}{\not{k} - m_t} \left(\frac{-iy_t^*}{\sqrt{2}} \frac{i}{\not{k} - m_t}\right)\right] . \qquad (1.27)$$

If we use the hard-momentum cut-off, this term will evaluate to become,

$$\delta m_H^2|_{\text{top}} = -\frac{N_c|\lambda_t|^2}{8\pi^2}\left[\Lambda^2 - 3m_t^2 \log\left(\frac{\Lambda^2 + m_t^2}{m_t^2}\right) + \ldots\right] . \qquad (1.28)$$

This is where SUSY comes in. Suppose we have N new scalar particles ϕ_L and ϕ_R, which gives the Lagrangian,

$$\mathcal{L}_{\text{scalar}} = -\frac{\lambda}{2}H^2\left(|\phi_L|^2 + |\phi_R|^2\right) - H\left(\mu_L|\phi_L|^2 + \mu_R|\phi_R|^2\right) - m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2 , \quad (1.29)$$

then when $N = N_c$, $\lambda = |\lambda_t|^2$ the quadratic divergences from the fermionic loop in Eq. (1.28) will be cancelled and setting $m_t = m_L = m_R$ and $\mu_L^2 = \mu_R^2 = 2\lambda m_t^2$ then also cancels out the logarithmic divergences [6]. The key is that supersymmetry seems to give a way out of this and it is a very elegant solution.

The sad thing is, with our good friend the LHC we have so far found no evidence of supersymmetry at 500GeV. In fact, the models that involve coupling supersymmetry to the Standard Model in the most simple way, known as **Minimal Supersymmetric Standard Models (MSSM)**, have now been ruled out by the LHC experiments. So to say that you should study supersymmetry because it solves the hierarchy problem is no longer, in its simplest form, valid [7].

1.5 The reasons why you should do supersymmetry

Okay. So we are back to square one. Not all hope is lost though. What I would really want to convince you is this — supersymmetry is still worth learning. And I have come up with three reasons.

1. **Understanding QFT better.** Throughout the course I will emphasise how much supersymmetry gives us amazing control. The reason is thus — from the QFT courses you have already seen how messy QFT are, even just on the perturbative level. This is basically all we know how to do, and when the coupling goes strong we basically are forced to stop. Supersymmetric field theories are a subset of QFTs, but they

[7] This is the viewpoint I take. There are however still a lot of people that argue otherwise.

turn out to place strong restrictions on the dynamics of the theory and therefore make things tractable. They also highlight many concepts that turn out to be very useful in understanding to wider class of QFTs, such as *dualities*, *phase transitions* and more.

2. **SUSY and mathematics.** Supersymmetric theories turns out to have a lot of deep connections with mathematics. In particular, the study of topological index theorems is manifest in supersymmetric theories of low dimensions and it is in fact where many current geometry and topology research is on. If you are interested, you can have a quick look at Appendix B where I have illustrated how the localisation principle highlights this relationship with topology.

3. **SUSY and phenomenology.** Ultimately we want to describe the world. However, as we have illustrated, SUSY does not exist, at least to our best knowledge, at TeV scale where we had expected new physics to pop up. This is not the end though. There is still about 15 orders of magnitude between M_P and Λ_{LHC}, and one can perhaps hope that supersymmetry will show up at some point. I am a string theorist, and if you were to believe that string theory is the best way to explain the world (as far as we know, it is the only consistent quantum gravity theory), then it seems like supersymmetry is needed for string theory. The reason is two-fold — on a historical perspective, adding fermions to string theory automatically gives you local supersymmetry (supergravity). Non-supersymmetric string theories seem to be inconsistent [8]. On a practical perspective, superstring theories are a lot better understood precisely because of the control SUSY buys us. So it might still be true that SUSY is there, and out-of-touch of whatever experiments we can construct. There are thousands of literature out there on the exact mechanisms of supersymmetry breaking (and how it related to other problems in string theory) and we will briefly discuss this later.

All hope is not lost. And I would advise you to stay with me and have fun on this little journey where we explore the many fun things about supersymmetry.

2 Spinor Representations

We can ask ourselves what we will need to understand supersymmetric field theories. Since supersymmetry is a symmetry between bosons and fermions, it is prudent for us to understand how to write down fermionic fields. How do we write down fermionic representations of the Lorentz group? This is the subject of this section.

2.1 Projective representations and spinors

In quantum mechanics, we know that physical states live in a Hilbert space. This is clearly the same in quantum field theory — a QFT is based on quantum mechanics, and the states

[8] I must admit even as a working string theorist I know very little about non-supersymmetric string theories, so I am taking the communities' word on this.

in a QFT should be represented by a ray in a Hilbert space $\mathcal{H}$. Why a ray? The physical states in QFT are identified up to a c-number,

$$\Psi \sim c\Psi \; , \quad c \in \mathbb{C}^* \; , \tag{2.1}$$

and typically we normalise the state so the states are identified up to a phase.

What does this have to do with our discussion? Recall the argument in *Groups and Representations* last term — we have some symmetries in a physical theory, and we will need some way to mathematically describe them in QFT. This is done via groups and representations — symmetries are abstractly described by a group structure, and its effects on the space of states are described by its representation on that vector space. Notice that our space is set of rays in a Hilbert space — our representation space is actually not the Hilbert space, but the projectivised version on that. Instead of 'normal' or 'regular' representations, we should really look at something known as projective representations.

Definition 2.1. Let G be a group and V a finite-dimensional vector space over a field $\mathbb{F}$. A map $\rho : G \to GL(V)$ is a **projective representation** of G over $\mathbb{F}$ if there exists a mapping $\alpha : G \times G \to \mathbb{F}^*$ such that the following two properties hold:

(1) $\rho(x)\rho(y) = \alpha(x,y)\rho(xy) \; , \; \forall x, y \in G$.

(2) $\rho(1) = \mathrm{id}_V$.

The two conditions imply that α satisfies the following properties:

(i) $\alpha(x,y)\alpha(xy,z) = \alpha(y,z)\alpha(x,yz) \; , \; \forall x,y,z \in G$.

(ii) $\alpha(x,1) = \alpha(1,x) = 1 \; , \; \forall x \in G$.

Alternatively, one can define projective representations as a map $\rho : G \to PGL(V)$, where $PGL(V)$ is the projective linear group of V.

Our physical Hilbert space is intrinsically a projective space, so the phase $\alpha(x,y)$ cannot be eliminated in any way. This has quite a few mathematical consequences. In deriving the Lie algebra for any symmetry, we will actually arrive at the central extension of the Lie algebra [9],

$$[T_a, T_b] = i f^a{}_{bc} T_a + i f_{bc} 1 \; . \tag{2.2}$$

The second term is known as the **central charge** — this modifies the Lie algebra and admits new classifications of representations. Intrinsically projective representations can arise in two ways — either via the presence of this central charge, or by the fundamental group $\pi_1(G)$ of the Lie group. We will not pursue this general line of thought here, but the interested amongst you can have a look at [7, 8].

We will however focus our attention on the Lorentz group $SO(1,3)$. In particular, do

[9] This is a whole other subject and requires a lot of mathematical exposition.

intrinsically projective representations arise via algebra and/or topology? You probably remember from last term that we have this identity,

$$SO(1,3) \cong \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2} \; , \tag{2.3}$$

The Lorentz group is not simply connected. So it must have intrinsically projective representations. In particular, since a double loop that goes twice from 1 to Λ to $\Lambda\tilde{\Lambda}$ and back to 1 is contractible, let us write,

$$\left[U(\Lambda)U(\tilde{\Lambda})U^{-1}(\Lambda\tilde{\Lambda})\right]^2 = 1 \; , \tag{2.4}$$

and rearranging gives,

$$U(\Lambda)U(\tilde{\Lambda}) = \pm U(\Lambda\tilde{\Lambda}) \; . \tag{2.5}$$

The same is true for the Poincaré group. This has a very important consequence — this sign identification gives two kinds of states. The states with integer spin will not be affected, by the states with half-integer spin will have a sign change when going on 2π around the axis. This gives a **superselection rule** — we do not mix states of integer and half-integer spins.

Having a superselection rule is mathematically cumbersome. Turns out there is a way we can work with regular representations. This requires the lifting the group G to the central extension of the universal covering group of the classical symmetry group $\hat{G}$ — and it turns out that the projective representations will then be lifted up to regular representations of $\hat{G}$. The details are sketched out in Appendix C, and I would encourage you to have a look. The key idea, however, is to work with the central extended universal cover of $SO(1,3)$ — i.e. $\mathrm{Spin}(1,3) \cong SL(2,\mathbb{C})$. This is how we will construct representations of fermions — some of the representations (spinor representations) of $SL(2,\mathbb{C})$ will exactly give us what we want!

2.2 Spinors — the physics approach

So we need fermions — the basic unit turns out to be something known as spinors. How do we construct spinors representations? For the sake of clarity, we will work with $SL(2,\mathbb{C})$ and stick with four-dimensions for the time being. We will think about how to extend these structures to other dimensions in the next subsection.

Recall that at the Lie algebra level we have the identification,

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \; , \tag{2.6}$$

so we can label the representations of $\mathfrak{sl}(2,\mathbb{C})$ with two numbers (j_+, j_-). This is already treated in Groups and Representations last term [10]. The fundamental and anti-fundamental representations of $\mathfrak{sl}(2,\mathbb{C})$ are exactly the Weyl spinors, which under some $S \in SL(2,\mathbb{C})$ transform as,

$$\psi_\alpha \mapsto S_\alpha{}^\beta \psi_\beta \; , \tag{2.7}$$

[10] If you have no idea what is going on, please have a look at §5.1-5.2 of Andre's notes here.

$$\bar{\psi}_{\dot{\alpha}} \mapsto (S^*)_{\dot{\alpha}}{}^{\dot{\beta}} \psi_{\dot{\beta}} \ . \tag{2.8}$$

We call these **left-handed Weyl spinors** ($(j_+, j_-) = (\frac{1}{2}, 0)$) and **right-handed Weyl spinors** ($(j_+, j_-) = (0, \frac{1}{2})$) respectively. In particular, for a complex representation of a Lie group, we can find another representation by taking the conjugate, i.e. we find a matrix C such that $S^* = CSC^{-1}$, so the Weyl spinors are related by,

$$(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}} \ . \tag{2.9}$$

The invariant tensors in $SL(2, \mathbb{C})$ act as invariant tensors,

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \tag{2.10}$$

with the minus signs assigned such that $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta^\gamma_\alpha$. As discussed in the lectures, this allow us to build bilinears which properties we have checked in Q4 of the first problem sheet.

We can now construct vectors. To do this note that vector representations are the representations with $(j_+, j_-) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$. We will need the Pauli matrices,

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = (\mathbb{1}_2, \sigma^i)_{\alpha\dot{\alpha}} \ . \tag{2.11}$$

Now write a scalar related to a vector by $X = x_\mu \sigma^\mu$, then you can show that (exercise!) the spinor bilinear $\psi X \bar{\chi} = \psi^\alpha X_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$ is invariant under $SL(2, \mathbb{C})$. This means that $\psi\sigma^\mu\bar{\chi}$ is a vector. you can similarly construct $\bar{\sigma} = \epsilon\sigma^T\sigma^T$ with,

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (\mathbb{1}_2, -\sigma^i)^{\dot{\alpha}\alpha} \ . \tag{2.12}$$

and the vector is,

$$\bar{\chi}\bar{\sigma}^\mu\psi = \bar{\chi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\psi_\alpha \ . \tag{2.13}$$

Can we construct a different type of spinors? It turns out the representation $(j_+, j_-) = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ gives something known as **Dirac spinors**, which we can write as,

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \ . \tag{2.14}$$

Here, we introduce the **Dirac matrices**,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \implies \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{2.15}$$

Here $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$ with σ^i the Pauli matrices. We call the last line (after the arrow) a matrix representation that satisfies the **Clifford algebra**. Brushing all the mathematical details aside for now (see the following subsection), the matrix representations γ^μ gives the matrices,

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \ , \tag{2.16}$$

which satisfies the Lorentz algebra. The vector space that these matrices act on by,

$$\psi^\alpha \mapsto S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x) , \tag{2.17}$$

is precisely the (Dirac) spinor space. We will see how we can find spinor representations in other dimensions by understanding the structure of Clifford algebras in diverse dimensions.

Back to 4d, and we now have Weyl and Dirac spinors [11]. It is important for you, in this course, to know how to operator spinor algebra. Let me summarise these rules as follows.

1. Contract undotted indices $\searrow$, $\psi^\alpha\chi_\alpha = \psi\chi$.
2. Contract dotted indices $\nearrow$, $\bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi}$.
3. Treat all spinors as Grassmann-valued, $\psi^\alpha\psi^\beta = -\psi^\beta\psi^\alpha$.
4. Epsilon symmetry rules, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = \epsilon_{\beta\alpha}$.
5. Raising and lowering indices with epsilon in the front, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$ (similarly for dotted indices).
6. Contraction rule for sigmas, $(\bar{\sigma})^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}$.

You should perhaps have a go at the problem sheet again if you are unsure how to apply these rules.

2.3 Clifford algebras and spinors

How do we extend this discussion to other dimensions? It turns out that we will need the technology of Clifford algebras. The logic is as follows. Mathematically, we will first define something known as a Clifford algebra which is an associative algebra over some field $\mathbb{K}$ satisfying the relation,

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2B_{ij}\mathbb{1} , \tag{2.18}$$

where $\mathbb{1}$ is the unit in the algebra. Alternatively, we can treat Clifford algebras à la Eq. (2.15), and then generate the Clifford algebras in different dimensions by tensoring up Pauli matrices. The Lorentzian versions will need some extra work. We then properly classify the Clifford algebras in different dimensions, and we will see that the spin algebra is embedded in the even-graded part of the Clifford algebra (more on that later). Looking at the representations of this subalgebra in different dimensions will then give the different spinors.

I don't want taint the main discussion of the notes with all the algebraic details. Instead, I encourage you to read Appendix D for its gory details. A physicist's way of understanding Clifford algebras is reviewed in [9], and a mathematician's approach can be found in [10, 11].

3 CFTs and SCFTs

We have discussed the SCFT algebra in the lectures as an extension of the SUSY algebra. Let us try and understand what the algebra means in very brief terms.

[11] There are also Majorana spinors, characterised by

3.1 Conformal Field Theories

Here I will briefly review what a conformal field theory is. Some good references include [12, 13].

Consider the space $\mathbb{R}^{1,n-1}$ with flat metric $g_{\mu\nu}$ of signature (p,q) and the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The **conformal group** is the subgroup of coordinate transformations that leaves the metric invariant up to a scale factor,

$$g_{\mu\nu} \mapsto \Omega(x) g_{\mu\nu}(x) \ , \tag{3.1}$$

which are angle-preserving transformations. The infinitesimal coordinate transformations are generated by $x^\mu \mapsto x^\mu + \epsilon^\mu$, which gives,

$$ds^2 \mapsto ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\, dx^\mu dx^\nu \ , \tag{3.2}$$

where

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\,(\partial \cdot \epsilon)\, \eta_{\mu\nu} \ . \tag{3.3}$$

For the dimensions $d \geq 3$, we have to lowest order in x the generators listed in Table 3.1. The group then has dimension $\frac{1}{2}(d+2)(d+1)$. The conformal group for $\mathbb{R}^{m,n}$ is isomor-

ϵ^μ	x'	Operator	Name
a^μ	$x + a$	P_μ	Translation
$\omega^\mu{}_\nu x^\nu$	Λx	$M_{\mu\nu}$	Lorentz
λx^μ	λx	D	Dilatation
$b^\mu x^2 - 2x^\mu b \cdot x$	$\frac{x+bx^2}{1+2b\cdot x+b^2x^2}$	P_μ	Special Conformal

Table 3.1: Table of the generators of the conformal group in $d \geq 3$. The first two columns give the infinitesimal and full coordinate transformations respectively, and the last two columns give the corresponding operator/generator and its name.

phic to the group $SO(m+1, n+1)$; the inversion being an additional discrete generator not continuously connected to the identity. It is the isometry group of the lightcone in $d = m + n$ dimensions. The Lie algebra $\mathfrak{so}(m+1, n+1)$ is generated by the operators/generators listed in Table 3.1, with the non-Poincaré generators having the following non-trivial commutation relations,

$$[M_{\mu\nu}, K_\rho] = g_{\nu\rho} K_\mu - g_{\mu\rho} K_\nu \ , \tag{3.4}$$

$$[D, P_\mu] = P_\mu \ , \tag{3.5}$$

$$[D, K_\mu] = -K_\mu \ , \tag{3.6}$$

$$[K_\mu, P_\nu] = 2\,(\eta_{\mu\nu} D - M_{\mu\nu}) \ . \tag{3.7}$$

In particular, the dilatation generator D generates the abelian Lie subalgebra $\mathfrak{so}(1,1)$ and therefore all other generators have a weight under it. This weight is known as the **scaling dimension**, with translations and special conformal transformations having weights $+1$

and -1 respectively. In radial quantisation, the dilatation generator D acts as the Hamiltonian and the states living in the system will be characterised by its scaling dimension and its $SO(d)$ spin. Time translations are therefore generated by dilatation [14]. We are not going into the details here, but to summarise; the unitary representations of the conformal group are generated with the highest weight state (**primary state**) defined as

$$K_\mu \left|[L]_\Delta\right\rangle = 0 \tag{3.8}$$

with K_μ acting as the lowering operator. Similarly, the **descent states** are generated by acting the primary state with P_μ and Lorentz generators. For the details see for example [15].

Two things to note here. Firstly, the case for $d = 2$ is a bit more complicated. The conformal group of the Euclidean plane for example is the group $SO(3,1)$ of Möbius transformations. This is a finite group, although looking at the Lie algebra, we will see that there is an infinite set of conformal Killing fields, which gives an infinite number of independent constraints [12]. For the conformal group of $\mathbb{R}^{1,1}$ this is indeed infinite as a group.

Another subtlety comes from a 'common' misconception with Weyl transformations which has the form,

$$g_{\mu\nu} \mapsto \Omega(x) g_{\mu\nu} \ . \tag{3.9}$$

Here we note that a Weyl transformation is a physical change of the metric and has nothing to do with coordinate transformations, where as a conformal transformation is by definition a coordinate transformation. One has to be careful with such a distinction in quantisation in string theory.

3.2 Superconformal Symmetry

We now discuss superconformal symmetries in very brief terms. The symmetry algebra of SCFTs contain both the conformal algebra $\mathfrak{so}(d,2)$ and a supersymmetry algebra involving the supercharges Q. To complete the algebra one must also include the supercharges S, where the scaling dimensions of Q and S are $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Similarly to before, we can define the irreps of the superconformal algebra in the radial quantisation scheme with K_μ acting as the lowering operators and mirroring the construction of irreps of super-Poincaré group with the **superconformal primary** defined to be the state annihilated by K, S and $\bar{S}$. For the details, please refer to [15, 16].

Under construction

Working in progress — this will be updated with a new discussion.

4 Induced representations and Wigner's classification

Induced representations are a key part to Wigner's Classification. Here we provide a quick review of the method.

[12]This is the origin for the claim that the 2d conformal group is infinite, see §2.4 of [13] for a detailed discussion.

4.1 Set-up

We already know what happens when we want to restrict a representation of G to a subgroup H of G - this is how branching rules arise. The key idea of an induced representation is to do the inverse - to generate a representation for a bigger group G given a representation of a subgroup $H \subset G$.

Let us assume we are given a representation $\rho : H \to GL(V)$ where V is a vector space and H is a subgroup of G, where for $h \in H$,

$$v \overset{h}{\mapsto} \rho(h)v, \quad v \in V \ . \tag{4.1}$$

We consider the cosets G/H and represent the each coset with an element g_i such that the coset $[g_i]$ is defined as,

$$[g_i] = \{g \in G \,|\, g_i = gh,\ h \in H\} \ . \tag{4.2}$$

Then, for any $g \in G$, we have,

$$gg_i = g_j h \tag{4.3}$$

for some $h \in H$. The number of cosets is N. We define the representation space as the product $G/H \times V$ with

$$v_i = ([g_i], v) \in G/H \otimes V \ , \tag{4.4}$$

such that

$$v_i = ([g_i], v) \overset{h}{\mapsto} ([g_j], \rho(h)v) = v_j \ , \tag{4.5}$$

under the action of $h \in H$. The representation space is therefore isomorohic to the N-fold tensor product $V^{\otimes N}$. In the space where we permute the N-copies of Vs, the representation matrices for the induced representation can then be given by $N \times N$ matrices with elements,

$$\rho_{ji}(g) = \begin{cases} \rho(h), & g_j^{-1} g g_i = h \in H \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

You can show that this construction indeed leads to a representation (it satisfies group homomorphism $\rho(g)\rho(g') = \rho(gg')$). The dimension of the induced representation of G is $N \times \dim \rho_H$. As a sanity check, if $H = \{e\}$, you should check that the induced representation is identical to the regular representation for finite groups. So the induced representation is in fact in general reducible.

4.2 An example - dihedral group D_n

Let us illustrate the construction of induced representations using the dihedral group D_n. Recall that the dihedral group D_n is generated by elements a and b where,

$$D_n = \left\{a, b \,\middle|\, a^n = b^2 = e, \quad ab = ba^{n-1}\right\} \ . \tag{4.7}$$

Let's choose H to be the abelian subgroup $\mathbb{Z}_n$ generated by the elements a, so we have one-dimensional representations labelled by k with

$$v \overset{a}{\mapsto} e^{\frac{2\pi k i}{n}} v \ . \tag{4.8}$$

The next step is to look at the cosets $D_n/\mathbb{Z}_n$. In fact there are two distinct ones labelled by $i = 1, 2$ where we take $g_1 = e$ and $g_2 = b$ to be the representatives. We see that under the action of a in D_n the elements $v_1 = (e, v)$ and $v_2 = (b, v)$ will transform as

$$(v_1, v_2) \overset{a}{\mapsto} \left(e^{\frac{2\pi k i}{n}} v_1, e^{-\frac{2\pi k i}{n}} v_2\right) , \tag{4.9}$$

but note that since acting b changes the coset but the corresponding element $h = g_j^{-1} g g_i$ does not exist the representation only acts on the first coordinate, leaving us with [13]

$$(v_1, v_2) \overset{b}{\mapsto} (v_2, v_1) . \tag{4.10}$$

Let us write $\vec{v} = (v_1, v_2)$ to be the representation space $V^{\otimes 2}$. Then we have,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \overset{a}{\mapsto} A_k \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi k i}{n}} & 0 \\ 0 & e^{-\frac{2\pi k i}{n}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} , \tag{4.11}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \overset{b}{\mapsto} B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} . \tag{4.12}$$

We see that the matrices $A_k^n = \mathbb{1}$, $B^2 = \mathbb{1}$ and $A_k B = B A_k^{n-1}$. This gives a two-dimensional representation of D_n for each k. Now we consider the following cases.

<u>n odd</u>

In this case we see that $k = 1, ..., \frac{n-1}{2}$ gives two inequivalent two-dimensional irreps. For $k = 0$, we have $A = \mathbb{1}_2$ so we can apply an orthogonal transformation to diagonalise B - giving two 1d irreps. This is listed in Table 4.1.

Irrep	Case	Rep of $(a^r, a^r b)$
$R_{1,1}$	$k = 0$	$(1, 1)$
$R_{1,2}$	$k = 0$	$(1, -1)$
$R_{2,k}$	$k = 1, ..., \frac{n-1}{2}$	$(A_k^r, A_k^r B)$

Table 4.1: Irreps of D_n constructed from induced reps method for odd n.

<u>n even</u>

Same as in the odd case, we see that $k = 1, ..., \frac{n-1}{2}$ gives two inequivalent two-dimensional irreps. For $k = 0$, we have $A = \mathbb{1}_2$ so we can apply an orthogonal transformation to diagonalise B - giving two 1d irreps. There is an additional case with $k = \frac{n}{2}$ where $A = -\mathbb{1}_2$ so we have two more 1d irreps. All of the irreps are listed in Table 4.2. Indeed, the number of representations match the number of conjugacy classes. We can even choose

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix} \tag{4.13}$$

to get

$$R A_k R^{-1} = \begin{pmatrix} \cos\left(\frac{2\pi k}{n}\right) & -\sin\left(\frac{2\pi k}{n}\right) \\ \sin\left(\frac{2\pi k}{n}\right) & \cos\left(\frac{2\pi k}{n}\right) \end{pmatrix}, \quad R B R^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.14}$$

[13] If you need more steps then see $v_1 = (e, v) \overset{b}{\mapsto} (b, v) = v_2$ and vice versa.

Irrep	Case	Rep of $(a^r, a^r b)$
$R_{1,1}$	$k = 0$	$(1, 1)$
$R_{1,2}$	$k = 0$	$(1, -1)$
$R_{1,3}$	$k = \frac{1}{2}$	$((-1)^r, (-1)^r)$
$R_{1,4}$	$k = \frac{1}{2}$	$((-1)^r, -(-1)^r)$
$R_{2,k}$	$k = 1, ..., \frac{n-1}{2}$	$(A_k^r, A_k^r B)$

Table 4.2: Irreps of D_n constructed from induced reps method for even n.

4.3 The Poincaré group

Now we can review the classification of irreducible unitary positive energy representation $\mathcal{H}$ of the Poincaré group P^n. The construction is first discussed by Wigner [**?**] and we provide a quick summary here. We set $V = \mathbb{R}^{1,n-1}$ be a vector space with linear coordinates $x^0, ..., x^{n-1}$ and the Lorentzian metric,

$$g = (dx^0)^2 - (dx^1)^2 - ... - (dx^{n-1})^2 \ . \tag{4.15}$$

We set $x^0 = ct$, where c is the speed of light. **Minkowski spacetime** M^n is the affine space which underlies V. The **lightcone** is the cone traced out by the vectors with zero norm known as **lightlike** vectors. Vectors inside and outside the lightcone are termed **timelike** and spacelike respectively and have positive and negative norms respectively. The group of isometries of V is the orthogonal group $O(1, n-1)$, which has four components distinguished by the determinant ± 1 and whether the forward lightcone is mapped to itself (or to the backward one). The identity component is known as the proper orthochronous Lorentz group $SO^+(1, n-1)$ and has a double cober $\mathrm{Spin}(1, n-1)$. The group of isometries of M^n includes the subgroup of translations T of V, and that quotienting by T is isomorphic to $O(1, n-1)$. The **Poincaré group** P^n is the double cover of the identity component of the group of isometries, fitting into the exact sequence,

$$0 \to T \to P^n \to \mathrm{Spin}(1, n-1) \to 0 \ . \tag{4.16}$$

We can alternatively write,

$$P^n = \mathrm{Spin}(1, n-1) \ltimes T \tag{4.17}$$

4.4 Wigner's Classification

Irreducible unitary representations of the Poincaré group are uniquely characterised by two parameters, mass and helicity, so we consider the two one-by-one.

<u>**Mass**</u>

First restrict the representation to the translation subgroup T. The restricted representation of $T \hookrightarrow P^n$ decomposes into a direct sum of irreducible unitary representations. Since T is abelian, these unitary irreps are one-dimensional and are specified by characters,

$$\chi : T \to U(\mathbb{C}) = S^1 \ . \tag{4.18}$$

We label these one-dimensional irreps by a dual four-vector $p^\mu \in T^*$ known as a **4-momentum**,

$$\chi_p(x) = e^{ip\cdot x}, \quad p \in T^* \tag{4.19}$$

The **mass** is defined as the magnitude of the four-momentum vector,

$$m^2 = p \cdot p \; . \tag{4.20}$$

In other words, the unitary irreps are defined by the elements of the Pontryagin dual of T. The action of P^n on the translation group T can be deduced via the Pontryagin isomorphism,

$$\chi_{p\cdot g} = \chi_p \circ g \tag{4.21}$$

which sends $p \mapsto p \cdot g$ of the same mass.

Spin and helicity

Now restrict the irrep $\mathcal{H}$ of P^n to a rep of V, which decomposes into a direct sum of one-dimensional representations indexed by 4-momenta.

$$\mathcal{H} = \int_{p\in W} V_p \; , \tag{4.22}$$

here W indexes the set of infinitesimal characters p permuted by the action of $SO^+(1, n-1)$ on V^*, following the discussion $p \mapsto p \circ g$. The representation $\mathcal{H}$ is irreducible, so p form an orbit of the action (of the 'Lorentz group'). Note that m^2 is constant on each orbit, and there are two orbit types - **massless** $m = 0$ and **massive** $m > 0$ representations. In two-dimensions the massless reps further break to **left-moving** and **right-moving** reps. This transitive action on W defines a connected groupoid with objects $p \in W$ and morphisms $p \mapsto p \cdot g$. The map $p \mapsto V_p$ then defines a linear representation of this groupoid and the action of P^n on irrep $\mathcal{H}$ can be determined with this groupoid representation.

The connectedness of the groupoid means the representation is determined by the restriction to any of the automorphism groups $\mathrm{Aut}(p)$ in the groupoid, so we have the identification of W as a homogeneous space,

$$W = P^n/\mathrm{Aut}(p) \; , \tag{4.23}$$

so then V acts trivially on V_p. Define the **little group**, L_p, as

$$0 \to V \to \mathrm{Aut}(p) \to L(p) \to 1 \tag{4.24}$$

such that

$$\mathrm{Aut}(p) = L_p \ltimes V \; . \tag{4.25}$$

The little group $L_p \subset \mathrm{Spin}(1, n-1)$ is the reductive part of the compact stabiliser subgroup of p. Since the representation $\mathcal{H}$ of P^n is obtained by constructing a homogeneous complex hermitian vector bundle over the orbit, which can be extended by the direct integral construction on the groupoid representation if the action of L_p on V_p is known. This is otherwise known as the method of induced representations. For $\mathcal{H}$ to be irreducible, the rep on L_p must also be an irrep. Therefore our problem reduces to finding irreps of the little group L_p in the two types or orbits.

The mathematical basis of the construction is the functional-equivalent construction for the induced representation method illustrated for finite groups. Here in particular we focus on unitary representations and therefore the representation space $\mathcal{H}$ is the space of integrable sections, a subspace of the total space of sections of the bundle $P \times_\rho V \to G/H$ where G is the Poincaré group P^n and H is the Lorentz group. The little group method ultilises the following construction. We denote U to be the unitary induced representation of P^n on the Hilbert space $\mathcal{H}$ of integrable sections and consider the restriction of U to the abelian subgroup T. Let $t = \exp a \cdot T$ with z_0 denoting the identity coset, then,

$$U_t \cdot \psi(z_0) = D(t) \cdot \psi(z_0) = \alpha(t) \cdot \psi(z_0) \ , \tag{4.26}$$

where $\alpha(t) = \exp a \cdot \alpha_*(T) = e^{ia\cdot p_0}$. Then,

$$\left(U_t \cdot \psi\right)(z) = e^{ia\cdot\Lambda(\sigma(z)^{-1})\cdot p_0} \cdot \psi(z) \ , \tag{4.27}$$

with Λ being the adjoint representation of some element of the Lorentz group. Here σ indicates the section and z a point in the homogeneous space $\mathrm{Spin}(1, n-1)/L(\alpha)$, where $L(\alpha)$ is the little group. Define $p = \Lambda(\sigma(z)^{-1}) \cdot p_0$, and we have identified between the points $z \in \mathrm{Spin}(1, n-1)/L(\alpha)$ and the orbit of p_0 under the adjoint action of $\mathrm{Spin}(1, n-1)$. In particular since p_0 is the fixed point of this orbit the identification is one-to-one. Therefore, each representation of T occurring in U will be uniquely indexed by the points in $\mathrm{Spin}(1, n-1)/L(\alpha)$, $z \leftrightarrow p$. The Mackey direct integral decomposition of a representation gives a more functional-analytical way of looking at this identification. We break up, similar to the construction above, the Hilbert space of integrable sections as the direct integral,

$$\mathcal{H} = \int_{\mathrm{Spin}(1,n-1)/L(\alpha)} d\nu(p) \mathcal{H}_{(p)} \ , \tag{4.28}$$

where on each $\mathcal{H}_{(p)}$, U_t is $e^{ia\cdot p} \times 1$. This is the spectral theorem for a commuting family of self-adjoint operators - since $U(T)$ consists of $\dim(T)$-families of commuting unitary operators, this decomposes $\mathcal{H}$ over the spectrum with each individual component shown to be in 1-1 correspondence with points of $\mathrm{Spin}(1, n-1)/L(\alpha)$. We then show that this representation is irreducible by proving that any bounded linear operator commuting with the representation U of P^n is by Schur's Lemma a multiple of the identity.

The result of the above mathematical discussion implies that one can consistently obtain the unitary irreducible representations of the Poincaré group using the little group method. The irreps will be characterised by the quadratic Casimirs of P^2 and W^2, the latter of which defined as,

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \tag{4.29}$$

known as the **Pauli-Lubański pseudovector**. This characterises the spin/helicity of the irrep.

We now separate the discussion into two distinct cases.

Massive Case $m > 0$

Fix the mass as m. We can take the basepoint $p = (m, 0, ..., 0)$ which gives the stabiliser subgroup (little group) as $\mathrm{Spin}(n-1)$. A massive particle then corresponds to an irrep of $\mathrm{Spin}(n-1)$ as in the non-relativistic case, with the Hilbert space being

$$\mathcal{H} = L^2(\mathbb{R}^{n-1}, r) \ , \tag{4.30}$$

where r is an irrep of $\mathrm{Spin}(n-1)$. The **spin** of a representation r of $\mathrm{Spin}(n-1)$ is defined as follows. Fix a 2-plane in $\mathbb{R}^{n-1}$, and consider the double cover $\mathrm{Spin}(2) \subset \mathrm{Spin}(n-1)$ of rotations in that plane which fixes the 2-plane. The irrep r decomposes into a sum of one-dimensional irreps and $\mathrm{Spin}(2)$ as by $\lambda \mapsto \lambda^{2j}$ with $\lambda \in \mathrm{Spin}(2)$ and j a half-integer. The spin is then defined as the largest $|j|$ that occurs in the decomposition. When $n = 4$, $\mathrm{Spin}(3) \cong SU(2)$ so $|j|$ simply labels the irreps of $SU(2)$.

Massless Case $m = 0$

We consider the basepoint $(1, 1, 0, ..., 0)$. The stabiliser subgroup in this case is the double cover of orientation-preserving isometries of an $(n-2)$-dimensional Euclidean space. The helicity λ of the irrep is the label j associated to the action of $\mathrm{Spin}(2) \subset \mathrm{Spin}(n-2)$.

5 Supermultiplets

In this section we will review how supersymmetric multiplets (supermultiplets) are constructed. We have previously discussed how one-particle Lorentz irreducible representations are constructed. Our group theory knowledge on branching tells us that in general if we have a bigger group $G \subset \hat{G}$, any irreducible representations of $\hat{G}$ are in general reducible representations of the smaller group G. In this case, when we promote our Poincaré group to the super-Poincaré group (or on the algebra level, the Poincaré algebra to the super-Poincaré algebra), irreps of super-Poincaré will become reducible representations of the Poincaré algebra. This leads to the following definition.

Definition 5.1. A **supermultiplet** or a **superparticle** is a representation of the super-Poincaré algebra which is a collection of one-particle Poincaré irreps.

In the lectures we have seen how the Coleman-Mandula theorem can be evaded by including graded Lie algebras which are Lie algebras that involve anti-commutators. This allows us to extend the Poincaré algebra to the super-Poincaré algebra, defined by,

$$[P_\mu, Q^I_\alpha] = 0 = [P_\mu, \bar{Q}^I_{\dot{\alpha}}] \ , \tag{5.1}$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \ , \quad [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \ , \tag{5.2}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \ , \tag{5.3}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \ , \quad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^* \ . \tag{5.4}$$

From this we can immediately deduce the following observations.

1. **Particles in the same multiplet have the same mass but different spins.** The two Casimirs of the Poincaré algebra, namely the mass and spin/helicity,
$$P^2 = P^\mu P_\mu \ , \quad W^2 = W^\mu W_\mu \ , \tag{5.5}$$
where the Pauli-Lubanski vector is defined as,
$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \ , \tag{5.6}$$
are no longer both Casimirs in the super-Poincaré algebra. The W^2 fails to be a super-Poincaré Casimir, and therefore particles belonging to the same multiplet have the same mass but different spin. Supersymmetry generator, Q^I_α, is a fermionic generator and can be used to relate one-particle irreps in the same supermultiplet.

2. **The energy of a state is non-negative.** If we consider an arbitrary state $|\phi\rangle$, we see that from a short computation (and tracing over the I, J indices),
$$\langle\phi| \{Q^I_\alpha, \bar{Q}^I_{\dot\alpha} |\phi\rangle \, \delta^{II} = 2\sigma^\mu_{\alpha\dot\alpha} \langle\phi| P_\mu |\phi\rangle \, \delta^{II} = ||(Q^I_\alpha)^\dagger |\phi\rangle ||^2 + ||Q^I_\alpha |\phi\rangle ||^2 \geq 0 \ . \tag{5.7}$$
Summing over the indices α and $\dot\alpha$ gives,
$$4 \langle\phi| P_0 |\phi\rangle \geq 0 \ . \tag{5.8}$$

3. **There is an equal number of bosonic and fermionic degrees of freedom in a supermultiplet.** To see this, we define the fermion number operator $(-1)^{N_F}$ which we can take $N_F = 2s$. The fermion number operator satisfies the following property,
$$\{Q^I_\alpha, (-1)^{N_F}\} = 0 \ . \tag{5.9}$$
so we can trace over and get,
$$0 = \mathrm{tr}\left(-Q^I_\alpha (-1)^{N_F} \bar{Q}^J_{\dot\beta} + (-1)^{N_F} \bar{Q}^J_{\dot\beta} Q^I_\alpha\right) = \mathrm{tr}\left[(-1)^F \left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\}\right] \ , \tag{5.10}$$
Now we can use Eq. (5.3), trace over I, J and choose $P_\mu \neq 0$ to get $\mathrm{tr}(-1)^{N_F} = 0$ which gives the result. Another way of seeing this is to directly use the counting operator. We take the binomial equation,
$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n} \ , \tag{5.11}$$
and specialise to $a = -b = -1$ to get,
$$0 = \sum_{n=0}^{N} \binom{N}{n} (-1)^n \ . \tag{5.12}$$
This formula simply counts successive levels of the supermultiplet with alternating signs (for fermions and bosons). This shows that counting states with N levels with the fermionic signs gives zero [14].

[14] Notice here we only require successive levels to take opposite signs of each other – the overall sign which determines whether the fermion number operator $(-1)^{N_F}$ acts positively or negatively is irrelevant.

4. **There is an outer automorphism of the super-Poincaré algebra called the R-symmetry.** In general, the supersymmetry generators and the internal symmetry generators will have non-zero commutation relations. The largest possible internal symmetry group which can act non-trivially on the supercharges is known as the **R-symmetry group**. It is an outer automorphism — an isomorphism of the super-Poincaré algebra where the generators are not in the original algebra itself. We will see later how R-symmetries play an important role in supersymmetry.

5. **There is a central extension of the super-Poincaré algebra.** The super-Poincaré algebra admits a central extension, where the generators are called central charges. These generators commute with the whole of the supersymmetry algebra and within themselves — they lie in the centre of the algebra. We can think of them as some linear combinations of the internal symmetry group generators of the compact super-Poincaré algebra and are quantum operators which gives different values from state to state.

Let us review how massless and massive multiplets are constructed in SUSY. Because particles have the same mass but different spin in a supersymmetric multiplet, we can just look at how the supersymmetry generator Q^I_α act on one-particle irreps and relate irreps in the same multiplet. Our analysis will therefore begin with the form of the supersymmetry generator Q^I_α and effectively treat components of the generator as 'Fock space generators'.

5.1 Massless multiplets

In some sense it is easier to start with massless multiplets. To do this, we go to the rest from of the massless particle where the four-momentum of the particle is $P^\mu = (E, 0, 0, E)$. Then,

$$\sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix} , \tag{5.13}$$

and therefore we see that,

$$\left\{ Q^I_\alpha, \bar{Q}^J_{\dot{\beta}} \right\} = \left[\begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix} \right]_{\alpha\dot{\beta}} \delta^{IJ} , \tag{5.14}$$

where we have assumed there are no central charges $Z^{IJ} = 0$. From this we see that the first component of the SUSY generator is trivially realised $\left\{ Q^I_1, \bar{Q}^J_{\dot{1}} \right\} = 0$. Since the Hilbert space is positive, we can sandwich this result between a ket and a bra $|\phi\rangle$ which implies,

$$||Q^I_1 |\phi\rangle ||^2 + ||\bar{Q}^I_{\dot{1}} |\phi\rangle ||^2 = 0 , \tag{5.15}$$

so we get the solution $Q^I_1 = \bar{Q}^J_{\dot{1}} = 0$. This means that half of the generators are killed. Let us redefine the remaining generators using,

$$a^I = \frac{1}{\sqrt{4E}} Q^I_2 , \quad (a^I)^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot{2}} . \tag{5.16}$$

They satisfy the relation,

$$\{a^I,\,(a^J)^\dagger\} = \delta^{IJ}\ , \tag{5.17}$$

which is just the Clifford algebra of N fermionic harmonic oscillators! To proceed, we can analyse the multiplets as representations of the Clifford algebras — this simply requires choosing starting points of the representation (vacuum) annihilated by all lowering operators a^I which will allow us to generate different supermultiplets using the set of raising operators $(a^I)^\dagger$.

In supersymmetry we define the starting point of the irrep as a Clifford vacuum.

Definition 5.2. A **Clifford vacuum** is a one-particle irreducible state annihilated by all lowering operators a^I as well as being eigenstates of the momentum operator,

$$P_0\,|\Omega\rangle = -im\,|\Omega\rangle\ ,\quad P_i\,|\Omega\rangle = 0\ , \tag{5.18}$$

$$a^I\,|\Omega\rangle = 0\ ,\quad \langle\Omega|\Omega|\Omega|\Omega\rangle = 1\ . \tag{5.19}$$

It is labelled by an integer which is half the spin of the lowest spin state in the multiplet $j = n/2,\ n \in \mathbb{N}_0$.

Let us understand why the Clifford vacuum $|\Omega\rangle$ can be labelled by the spin number j. Recall in Wigner's classification massless irreps of the Lorentz algebra can be assigned to an irrep of $ISO(2)$, or, more precisely, ignoring the continuous spin representations, an irrep of $SO(2)$. This means that it is an eigenstate of the helicity operator J_3 [15]. There is also an $SU(N)\times U(1)$ group of automorphisms, with the generators of $SU(N)$ and $U(1)$ given by,

$$T_{IJ} = \frac{1}{2}(a_I^\dagger a_J - a_J^\dagger a_I) - \frac{1}{2N}\delta_{IJ}(a_K^\dagger a_K - a_K a_K^\dagger)\ , \tag{5.20}$$

$$W = \frac{1}{4}[a_K^\dagger, a_K]\ . \tag{5.21}$$

The action of W coincides with J_3, as,

$$[W, a_I^\dagger] = -\frac{1}{2}a_I^\dagger\ . \tag{5.22}$$

In light of this, the Clifford vacuum $|\Omega\rangle$ is therefore assigned to be an irrep of $SO(2)\times SU(N)$ — it has a helicity λ and some representation R of $SU(N)$. Here the $SU(N)$ is precisely the R-symmetry, which is an outer automorphism of the supersymmetry algebra. Typically we take $R = 1$, i.e. we set the R-symmetry rep to be the trivial rep for the Clifford vacuum.

Note that a Clifford vacuum is not a true vacuum. A vacuum state in a QFT is the lowest energy state in that theory — here the Clifford vacuum is simply defined as the state annihilated by all the lowering operators, which is not directly related to operators which lower energy quanta in our system.

[15] Remember the discussion of the projective representations mean this group should be completed to its universal cover.

Upon picking a Clifford vacuum of a certain spin, say $|\Omega_0\rangle$, we can now use the raising operators to obtain the full multiplet. The raising operators $a_I^\dagger$ is a representation $(\frac{1}{2}, \square)$ in $SO(2)\times SU(N)$, so therefore it raises the helicity by $\frac{1}{2}$ whilst tensoring the representation R with fundamental representations (antisymmetrised with respect to the raising operators). Here we have two immediate comments.

1. **CPT-invariant multiplets.** In physics we would like to look at representations of the algebra that are CPT-invariant. However, since CPT flips the sign of the helicity, the supermultiplet is not immediately CPT-invariant unless the helicity of the multiplet content is centred around $j = 0$. This means that we will need to double the entire supermultiplet with a CPT-conjugate one to get a CPT-invariant multiplet.

2. **Maximum spin of the Clifford vacuum.** For interacting QFTs, recall that the spins of the particles do not exceed $j = 1$. For interacting theories with gravity the spins of the particles do not exceed $j = 2$. This puts a limit to the number of possible Clifford vacua that we can use the raising operators on.

Example 5.1. To illustrate this idea, let us look at the full 4d $\mathcal{N} = 1$ supermultiplet generated from Ω_0. Here we only have one raising operator, $a^\dagger$, so the states are,

$$\begin{aligned} |\Omega_0\rangle \;&\leftrightarrow \text{ complex scalar} \\ a^\dagger |\Omega_0\rangle \;&\leftrightarrow \text{ Weyl fermion} \end{aligned} \tag{5.23}$$

Notice that this state is not CPT-invariant. To obtain the full matter multiplet, we must add in the CPT-conjugate version of this which has the Weyl fermion with spin $j = -\frac{1}{2}$. The degrees of freedom of this supermultiplet contain one Weyl fermion and one complex scalar, and this supermultiplet is known as the **matter (chiral) multiplet** or the **Wess-Zumino multiplet**, i.e.

$$\Omega_0 \longrightarrow \left(0, \frac{1}{2}\right) \oplus \left(-\frac{1}{2}, 0\right) \;. \tag{5.24}$$

For the gauge multiplet, we start with $\Omega_{1/2}$, which gives,

$$\begin{aligned} \left|\Omega_{1/2}\right\rangle \;&\leftrightarrow \text{ Weyl fermion} \\ a^\dagger \left|\Omega_{1/2}\right\rangle \;&\leftrightarrow \text{ vector boson} \end{aligned} \tag{5.25}$$

So together with the CPT-conjugate we will get the **vector multiplet**,

$$\Omega_{1/2} \longrightarrow \left(\frac{1}{2}, 1\right) \oplus \left(-1, -\frac{1}{2}\right) \;. \tag{5.26}$$

The chiral and vector multiplets are the only multiplets with all field contents with spin smaller than 1. This is to ensure we have non-trivial interacting theories in 4d.

Notice that we have set the central charges to be trivial $Z^{IJ} = 0$. The reason for that is straightforward – in the presence of central charges the supermultiplet must obey the BPS bound given by $2m \geq Z^{IJ}$ (which we will see later), and since $m = 0$ for the massless case the central charge is always trivially realised.

Multiplet content as R-symmetry representations. As mentioned above, since R-symmetry is an internal symmetry that effectively rotates the fermionic charges into one-another, the states in a massless supermultiplet can be labelled by irreps of R-symmetry. If we take the convention that the Clifford vacuum is labelled by a trivial representation, then the massless multiplet is just antisymmetric irreps of R-symmetry, with the number of indices dependent on the number of raising operators.

To illustrate this let us look at the $\mathcal{N} = 4$ massless supermultiplet generated from the Clifford vacuum $|\Omega_{-1}\rangle$ as an example. The states generated by the operators

$$\left(a_1^I\right)^\dagger = \frac{1}{\sqrt{4E}} \left(Q_1^I\right)^\dagger \tag{5.27}$$

will be the multiplet as represented schematically in Table 5.1. For example, we can write

state (schematic)	helicity	$\mathbf{R}$	tensor
$\vert\Omega_{-1}\rangle$	-1	$\mathbf{1}$	T
$a^\dagger \vert\Omega_{-1}\rangle$	$-\frac{1}{2}$	$\mathbf{4}$	T_I
$a^\dagger a^\dagger \vert\Omega_{-1}\rangle$	0	$\mathbf{6}$	T_{IJ}
$a^\dagger a^\dagger a^\dagger \vert\Omega_{-1}\rangle$	$\frac{1}{2}$	$\overline{\mathbf{4}}$	T^I
$a^\dagger a^\dagger a^\dagger a^\dagger \vert\Omega_{-1}\rangle$	1	$\mathbf{1}$	T

Table 5.1: A list of the states, their helicities, together with their irrep in the R-symmetry group $SU(4)$ for the $\mathcal{N} = 4$ massless supermultiplet. Here I have represented the state using tensor methods in representation theory. The states are schematically illustrated and do not actually refer to the actual form of the state.

the states of the form $a^\dagger |\Omega_{-1}\rangle$ in the fundamental of $SU(4)$, the R-symmetry group, as a vector,

$$a^\dagger |\Omega_{-1}\rangle = \begin{pmatrix} \left(a_1^1\right)^\dagger |\Omega_{-1}\rangle \\ \left(a_1^2\right)^\dagger |\Omega_{-1}\rangle \\ \left(a_1^3\right)^\dagger |\Omega_{-1}\rangle \\ \left(a_1^4\right)^\dagger |\Omega_{-1}\rangle \end{pmatrix} \tag{5.28}$$

with the subscript labelling the extended SUSY label. The state then transforms under R-symmetry in the fundamental $a^\dagger |\Omega_{-1}\rangle \to U a^\dagger |\Omega_{-1}\rangle$ where $U \in SU(4)$. This is represented by the Young table:

$$\square$$

The second level is generated by two anticommuting operators and therefore must furnish the antisymmetric representation represented by the Young diagram

$$a^\dagger a^\dagger |\Omega_{-1}\rangle \longleftrightarrow \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \tag{5.29}$$

so hence the irrep $\mathbf{6}$. This generalises to multiplets in other extended supersymmetric theories.

5.2 Massive multiplets

Now we come to the difficult task of dealing with massive multiplets. For massive multiplets the extra complication comes from the presence of central charges. The analysis is in fact very similar to the massless case. Let us first look at the case where there are no central charges, $Z^{IJ} = 0$. In this case, we pick the four-momentum vector to be the one in the rest frame, $P_\mu = (m, 0, 0, 0)$, which now gives,

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}}\delta^{IJ} \ . \tag{5.30}$$

Compared to the massless case, we now have two sets of generators that can generate fermionic Fock states. In particular we can define the generators using,

$$a^I_{(1,2)} = \frac{1}{\sqrt{2m}} Q^I_{(1,2)} \ , \quad (a^I_{(1,2)})^\dagger = \frac{1}{\sqrt{2m}} \bar{Q}^I_{(\dot{1},\dot{2})} \ , \tag{5.31}$$

where the subscripts indicate the two components of the generators. The generators now satisfy the relation,

$$\{a^I_\alpha, (a^J_{\dot{\beta}})^\dagger\} = \delta^{IJ}\delta_{\alpha\dot{\beta}} \ . \tag{5.32}$$

This is the Clifford algebra of $2N$ fermionic harmonic oscillators. We can now proceed as in the massless case — choose a Clifford vacuum, successively generate the states with raising operators, and obtain the full multiplet by looking at the CPT-transformed ladder. Note that the Clifford vacuum can be assigned as an irreducible representation of $SU(2) \times SU(N) \times U(1)$ where $SU(N)$ is the little group for the massless states for Lorentz group and $SU(N) \times U(1)$ is the automorphism symmetry group.

Let us see in a bit more detail how this works. The state generated by acting n raising operators given by,

$$a^{I\dagger}_\mu \dots a^{J\dagger}_\nu |\Omega\rangle \tag{5.33}$$

is completely antisymmetric with respect to the exchange of the pair of indices $(I, \mu) \leftrightarrow (J, nu)$. This means that when the symmetry of the indices μ and I are represented by Young tableau, they are associated Young tableau (related to each other by flipping along a 45° angle). In particular, a Young diagram with p 2-high columns and $n - p$ 1-high column will give the spin of the state,

$$J = \frac{n}{2} - p \ , \tag{5.34}$$

and the dimension of the irrep of the same state in the multiplet in $SU(N)$ can be determined using Young diagram methods. The generators of $SU(N)$ are given by,

$$T_{IJ} = \frac{1}{2}(a^I_\mu a^{J\dagger}_\mu - a^{J\dagger}_\mu a^I_\mu) - \frac{2}{N}\delta_{IJ} W \ , \tag{5.35}$$

where W is the $U(1)$ generator,

$$W = \frac{1}{2} a^{I\dagger}_\mu a^I_\mu - \frac{N}{2} \ . \tag{5.36}$$

One can check that the eigenvalue of W is $(n-N)/2$ so it is the number operator counting how many $a^\dagger$ has been applied to the Clifford vacuum.

Let us illustrate this using the following example.

Example 5.2. Consider $\mathcal{N} = 2$ SUSY and start with a Clifford vacuum with $j = 0$. We have,

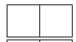
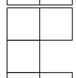
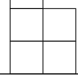
Level	States	j	$SU(2)$ YD
0	$\lvert\Omega_0\rangle$	0	·
1	$(a^1_\mu)^\dagger \lvert\Omega_0\rangle$, $(a^2_\mu)^\dagger \lvert\Omega_0\rangle$	$\frac{1}{2}$	
2	$\epsilon^{\mu\nu}(a^1_\mu)^\dagger(a^1_\nu)^\dagger \lvert\Omega_0\rangle$, $\epsilon^{\mu\nu}(a^1_\mu)^\dagger(a^2_\nu)^\dagger \lvert\Omega_0\rangle$, $\epsilon^{\mu\nu}(a^2_\mu)^\dagger(a^2_\nu)^\dagger \lvert\Omega_0\rangle$	0	
2	$(a^1_1)^\dagger(a^2_1)^\dagger \lvert\Omega_0\rangle$, $(a^1_2)^\dagger(a^2_2)^\dagger \lvert\Omega_0\rangle$, $\left[(a^2_2)^\dagger(a^1_1)^\dagger + (a^1_2)^\dagger(a^2_1)^\dagger\right] \lvert\Omega_0\rangle$	1	
3	$(a^2_\lambda)^\dagger\epsilon^{\mu\nu}(a^1_\mu)^\dagger(a^1_\nu)^\dagger \lvert\Omega_0\rangle$, $(a^1_\lambda)^\dagger\epsilon^{\mu\nu}(a^2_\mu)^\dagger(a^2_\nu)^\dagger \lvert\Omega_0\rangle$	$\frac{1}{2}$	
4	$\epsilon^{\lambda\sigma}(a^2_\lambda)(a^2_\sigma)^\dagger\epsilon^{\mu\nu}(a^1_\mu)^\dagger(a^1_\nu)^\dagger \lvert\Omega_0\rangle$	0	

Table 5.2: A list of the states and the corresponding level together with their spins for the $\mathcal{N} = 2$ SUSY $j = 0$ supermultiplet. The $SU(2)$ Young diagrams are also given.

Massive multiplets with non-zero central charges. So far we have considered cases where $Z^{IJ} = 0$. We have however, shown that the extended SUSY algebras can be extend by adding a central charge.

$$\{Q^I_\alpha, Q^{J\dagger}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu\delta^{IJ} \ , \tag{5.37}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta}Z^{IJ} \ , \tag{5.38}$$

$$\{Q^\dagger_{\dot\alpha I}, Q^\dagger_{\dot\beta J}\} = 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}Z^*_{IJ} \ , \tag{5.39}$$

with $\epsilon = i\sigma^2$ as before. The central charge matrix Z^{IJ} is antisymmetric in I and J and can be skew-diagonalised to $\mathcal{N}/2$ eigenvalues using,

$$\tilde{Z}^{LM} = U^{LI}Z^{IJ}(U^{MJ})^\dagger \ , \tag{5.40}$$

for some unitary matrix $U \in U(\mathcal{N})$, with $\tilde{Z}$ being block diagonal of the form,

$$\tilde{Z}^{IJ} = \epsilon_{ab} \otimes \mathrm{diag}(Z_i) \ , \tag{5.41}$$

for even $\mathcal{N}$ and for odd ones with an extra row and column of zeros at the end. With this, the original fermionic generators no longer function as ladder operators, but instead we will need to define the new ladder operators by making a rotation [16],

$$A^n_\alpha = \frac{1}{\sqrt{2}}\left[Q^{n-1}_\alpha + \epsilon_{\alpha\beta}\left(Q^n_\beta\right)^\dagger\right] \ , \tag{5.43}$$

[16] Notice that these expressions are not Lorentz-invariant — we have already fixed a frame $P^\mu = (m, 0, 0, 0)$ and have therefore brkoen Lorentz invariance. A Lorentz-invariant expression can be similarly found, for example,

$$A_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha + \frac{1}{M}\epsilon_{\alpha\beta}P^\mu\sigma^{\mu,\beta\dot\gamma}\bar{Q}^{\dot 2}_{\dot\gamma}) \ . \tag{5.42}$$

$$B^n_\alpha = \frac{1}{2}\left[Q^{n-1}_\alpha - \epsilon_{\alpha\beta}\left(Q^n_\beta\right)^\dagger\right] , \tag{5.44}$$

with $n \in \{1, \ldots, \mathcal{N}/2\}$. This gives the algebra with non-trivial commutators

$$\{A^n_\alpha, A^{m\dagger}_\beta\} = \delta^{nm}\delta_{\alpha\beta}(M + \sqrt{2}Z_n) , \tag{5.45}$$

$$\{B^n_\alpha, B^{m\dagger}_\beta\} = \delta^{nm}\delta_{\alpha\beta}(M - \sqrt{2}Z_n) , . \tag{5.46}$$

Let us impose the condition that the irreducible representations of the algebra must be unitary with semi-definite positive norm, which requires the unit norm state $|M, Z\rangle$ labelled by mass M and central charge Z,

$$||B^\dagger_\alpha |M, Z\rangle || \geq 0 \Longrightarrow M \geq \sqrt{2}Z . \tag{5.47}$$

This is a very strong constraint known as the BPS bound. A few comments follow.

- For massless states $Z = 0$. This is exactly the reason mentioned in the subsection above why we don't need to consider non-zero Z in the massless case.
- For massive states that satisfies one or more of the n equalities $M = \sqrt{2}Z_n$, the multiplet is annihilated by half of the supercharges. We see that in this case the B^n_α generate null states similar to the massless case so the multiplet is reduced to a much smaller one. This is known as the **short multiplet** where its normal $M > \sqrt{2}Z_n$ counterpart is known as the **long multiplet**. For the extreme case where all n inequalities are satisfied, the multiplet is known as a **ultra-short multiplet**.
- The construction of the short multiplet follows the normal procedure of generating supersymmetric multiplets (except without using the null B^n_α operators). As an example, the $\mathcal{N} = 2$ massive multiplet with a non-zero charge has three states,

  $$|\Omega_0\rangle \ , \quad A^\dagger_\alpha |\Omega_0\rangle \ , \quad A^\dagger_1 A^\dagger_2 |\Omega_0\rangle \ . \tag{5.48}$$

  This is however not a CPT-invariant multiplet as it is a fermionic $SU(2)$ doublet which is a pseudo-real representation. By adding in the CPT conjugate, we can now show that this is the same as the CPT-invariant $\mathcal{N} = 2$ massless multiplet constructed from $\left|\Omega_{-\frac{1}{2}}\right\rangle$.

In particular the massive states where the bounds are saturated $M = \sqrt{2}Z$ are known as BPS states. These are the states with exactly half of the supersymmetry of the system, and are interestingly related to non-perturbative effects of the system [6].

Under construction

Working in progress — non-perturbative discussion will be updated.

6 Supersymmetric field theories; the component and superfield formalism

Ultimately, we want to study supersymmetric theories and the consequences of having supersymmetry in a theory. In supersymmetric theories there are two ways of writing down supersymmetric Lagrangians — the component formalism, which uses component fields; and the superfield formalism, which uses superfields that take coordinates from superspace. The two formalisms are entirely equivalent and the use of them is simply down to a matter of convenience. For example, in 4d $\mathcal{N} = 1$ supersymmetric theories, the $\mathcal{N} = 1$ superspace can be readily defined and therefore normally the superfield formalism is preferred.

In this section we will talk about the component and superspace formalism. I will aim to highlight the theoretical relationship between the two formalisms by introducing them in order. In doing so, we must also understand what a superspace is. The discussion here is a mathematical and physical summary of the component formalism, superspace, and the superfield formalism. A mathematical discussion can be found in Appendix.

6.1 Component formalism

Remember we have constructed supermultiplets as irreducible representations of the super-Poincaré algebra. Clearly, supersymmetric theories that involve one super-Poincaré irrep will have the field content precisely of one supermultiplet constructed using the methods in the previous section. The field content is the component fields.

Definition 6.1. A **component field** is a field, $\phi : M^{1,3} \to X$, that is part of the field content of a supermultiplet.

The **component formalism** is simply a way of writing down supersymmetric field theories using component fields of a supermultiplet. That is it. There is really nothing more to it.

The free Wess-Zumino model. Perhaps it is best to illustrate this with an example. Let us stick to 4d $\mathcal{N} = 1$ supersymmetric field theories, and consider the massless chiral multiplet which involves a spin-0 field and a spin-$\frac{1}{2}$ field. The corresponding theory therefore must have a scalar and a fermion. Let us think about what the simplest thing we can write down (that is complex and can potentially have a supersymmetry, a symmetry between bosons and fermions) — a free theory with a complex scalar field and a Weyl fermion:

$$S = \int d^4x \left(-\partial^\mu \phi^* \partial_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi \right) , \tag{6.1}$$

where we clearly have two parts to the Lagrangian - the scalar part is a bosonic complex scalar field,

$$\mathcal{L}_B = -\partial^\mu \phi^* \partial_\mu \phi , \tag{6.2}$$

whilst the simplest fermionic part is Weyl fermion,

$$\mathcal{L}_F = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi . \tag{6.3}$$

Suppose now we want to impose some supersymmetry, i.e. we want the system to have a symmetry that transforms between the two Lagrangians (c.f. Eq.(1.2)). For the bosonic part we must have

$$\delta\phi = \epsilon\psi \tag{6.4}$$

$$\delta\phi^* = \epsilon\psi^\dagger \ , \tag{6.5}$$

which gives,

$$\delta\mathcal{L}_B = -\epsilon\partial^\mu\psi\partial_\mu\phi^* - \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi \ . \tag{6.6}$$

We want the transformation of the fermionic Lagrangian to cancel this bosonic transformation up to a total derivative (as a total derivative will be cancelled out in $\int d^4x$ of the action). To do this we set,

$$\delta\psi_\alpha = -i\left(\sigma^\mu\epsilon^\dagger\right)_\alpha \partial_\mu\phi \ , \tag{6.7}$$

$$\delta\psi^\dagger_{\dot\alpha} = -i\left(\epsilon\sigma^\mu\right)_{\dot\alpha}\partial_\mu\phi^* \ , \tag{6.8}$$

giving,

$$\begin{aligned}\delta\mathcal{L}_F &= -\epsilon\bar\sigma^\mu\sigma^\nu\partial_\nu\psi\partial_\mu\phi^* + \psi^\dagger\bar\sigma^\nu\sigma^\mu\epsilon^\dagger\partial_\mu\partial_\nu\phi \ , \\ &= -\delta\mathcal{L}_B + \partial_\mu\left(\dots\right) \ .\end{aligned} \tag{6.9}$$

That looks good. Is this a supersymmetry though? In particular, we will need to check whether the supersymmetry algebra closes. Specifically to satisfy the SUSY algebra $\{Q,Q\}\sim P$, we must have,

$$\left(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}\right)X = -i\left(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger\right)\partial_\mu X \ . \tag{6.10}$$

We can check this for $X = \phi, \phi^*, \psi, \psi^\dagger$, but we see that,

$$\left(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}\right)\psi_\alpha = -i\left(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger\right)\partial_\mu\psi_\alpha + i\left(\epsilon_{1\alpha}\epsilon_2^\dagger\bar\sigma^\mu\partial_\mu\psi - \epsilon_{2\alpha}\epsilon_1^\dagger\bar\sigma^\mu\partial_\mu\psi\right) \ . \tag{6.11}$$

The last term vanishes on-shell, i.e. when the equation of motion,

$$\bar\sigma^\mu\partial_\mu\psi = 0 \ , \tag{6.12}$$

is satisfied, the last term vanishes. This is fine as long as we evaluate the system **on-shell** (when the equations of motion are satisfied).

Generically however in QFT we know that to evaluate scattering amplitudes the Feynman rules are derived off-shell, i.e. even when the equations of motion are not satisfied. Virtual particles do exist in our formalism, and we will need to cover these cases as well. This however suggests that supersymmetry breaks down in this case! To resolve this, we use a trick called **auxiliary fields** where we add in the term to the Lagrangian,

$$\mathcal{L}_{\text{aux}} = F^*F \ . \tag{6.13}$$

Now $F = 0$ holds on-shell, showing that these auxiliary fields are indeed sole mathematical devices to make everything consistent. We now demand that,

$$\delta F = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi \ , \tag{6.14}$$
$$\delta F^* = -i\partial_\mu \psi^\dagger \bar{\sigma}^\mu \epsilon \ , \tag{6.15}$$

then the SUSY algebra closes off-shell, where Eq. (6.10) is satisfied for $X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*$. This works because there was originally a mismatch of degrees of freedom - on-shell the fermionic equations of motion reduces the number of degree of motion by a factor of two [17]. The auxiliary fields hence fill the missing bosonic degrees of freedom so we have a match of degrees of freedom on both side in both the on-shell and off-shell cases (see Table 6.1). This

Field	Spin	On-shell d.o.f.	Off-shell d.o.f.
ϕ, ϕ^*	0	2	2
$\psi_\alpha, \psi^\dagger_{\dot{\alpha}}$	1/2	2	4
F, F^*	0	0	2

Table 6.1: Degrees of freedom in the free Wess-Zumino model in the on-shell and off-shell cases.

is known as the free Wess-Zumino model. As mentioned bofore, it gives the simplest supersymmetric multiplet — the chiral multiplet of four-dimensional $\mathcal{N} = 1$ supersymmetric theories. We will now learn how this can be derived from the superfield formalism.

6.2 Superspace

To discuss the superfield formalism, we must first discuss what is the superspace. In simple words, superspace is just the Minkowski space endowed with extra fermionic coordinates θ. There are many ways of defining a superspace:

- In the lectures it was introduced as a coset manifold — or in other words a homogenous space of the form G/H where G and H are Lie groups. This is known as the coset construction.
- One can also formalise superspace as a ringed space with Grassmann variables. This is related to the ringed space construction of manifolds in algebraic geometry, and it is a bit more mathematically involved.

Here we will only make a quick review of coset spaces and Grassman variables. A more detailed discussion of both can be found in Appendix.

[17] To see this, consider the frame where the fermion momentum is $p^\mu = (E, 0, 0, E)$ and so the equation of motion reads,

$$\bar{\sigma}^\mu p_\mu \psi = \begin{pmatrix} 0 & 0 \\ 0 & 2p \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ . \tag{6.16}$$

We see that then half of the fermionic degrees of freedom is projected out on-shell but off-shell it is an element of $\mathbb{C}^2$.

Coset spaces. Let us first quickly review the coset construction of superspaces. Coset space is a particular construction of homogeneous spaces in differential geometry, generated from quotients of Lie groups by closed subgroups $H \subset G$. Let us first recall a few definitions.

Definition 6.2. Suppose $\theta : G \times M \to M$ is a left action of a group G on the manifold M. For $p \in M$, we define,

- the **orbit of** p is the set of points generated by the action of all elements $g \in G$,

$$G \cdot p = \{g \cdot p \,|\, g \in G\} \; . \tag{6.17}$$

- the **stabiliser of** p is the set of group elements fixing g,

$$G_p = \{g \in G | g \cdot p = p\} \; . \tag{6.18}$$

The action θ is called

- **free** if the only element that fixes any $p \in M$ is the identity, i.e. G_p for any $p \in M$ is trivial.
- **transitive** if for $p, q \in M$ there exists $g \in G$ to take between them, $g \cdot p = q$.

Definition 6.3. A **homogeneous** G**-space** is a smooth manifold endowed with a smooth transitive action by a Lie group G.

Example 6.1. A simple example of a homogeneous space is S^{n-1} — it is an $O(n)$-homogenous space.

Given a Lie group G and its closed Lie group H, we can form left cosets of the form,

$$gH = \{gh | h \in H\} \; . \tag{6.19}$$

The quotient space is the left coset space G/H. This is a homogeneous G-space, as given by the following theorem [17].

Theorem 6.1. *Let G be a Lie group and let H be a closed subgroup of G. The left coset space G/H is a topological manifold of dimension equal to* $\dim G - \dim H$*, and has a unique smooth structure such that the quotient map $\pi : G \to G/H$ is a smooth submersion. The left action $g_1 \cdot (g_2 H) = g_1 g_2 H$ makes G/H into a homogeneous G-space.*

Proof. Given in Theorem 21.17 of [17]. □

We can analyse the coset space by looking at its Lie algebra which will give local coordinates to the coset manifold via the exponential map. The Lie algebra of a coset space G/H is decomposed into a direct sum:

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}^c \; , \tag{6.20}$$

with $\mathfrak{h}^c$ indicating the complement to $\mathfrak{h}$ in $\mathfrak{g}$. Given the generators of $\mathfrak{g}$, $[T_A, T_B] = f_{AB}{}^C T_C$, we can split the generators as $T_A = (M_I, K_a)$ where $M_I \in \mathfrak{h}$ and $K_a \in \mathfrak{h}^c$. A general element $g \in G$ and $h \in H$ can then be written as,

$$g = e^{i\epsilon^A T_A} = e^{i\eta^I M_I + i\alpha^a K_a} \ , \quad h = e^{i\tilde{\eta}^I M_I} \ , \tag{6.21}$$

with $\epsilon^A = (\eta^I, \alpha^a)$ and $\tilde{\eta}^I$ real coefficients. For G/H to be a coset space we require the coset to be **reductive**,

$$[\mathfrak{h}, \mathfrak{h}^c] \subset \mathfrak{h}^c \ , \tag{6.22}$$

and additionally it will be a symmetric space [18] if,

$$[\mathfrak{h}^c, \mathfrak{h}^c] \subset \mathfrak{h} \ . \tag{6.23}$$

Let us use this to study actions on the coset space. Firstly, write $\mathbf{x}(z) = e^{i\alpha^a K_a} \in G$ which labels the cosets. Now G/H is a homogeneous G-space — this means that G acts transitively on the set of cosets. This action can be written as,

$$g^{-1}\mathbf{x}(z) = \mathbf{x}(z')h(g, z) \ . \tag{6.24}$$

where z, z' labels some coordinate in G (namely, labelling the cosets). The transformation of the coordinates $z \mapsto z'$ can be deduced from using the BCS formula and working to first order.

The differential operators of the associated transformation can also be deduced. To see this, let us look at what happens to the transformation $g : z \mapsto z'$. By expanding about z to first order, $g \simeq 1 + i\epsilon^A T_A$ or,

$$z'^a \simeq z^a + \epsilon^A k^a_A(z) \ , \tag{6.25}$$

the differential operator,

$$\mathbf{T}_A = -ik^a_A(z)\frac{\partial}{\partial y^a} \ , \tag{6.26}$$

then realises the Lie algebra $\mathfrak{g}$ on the space of functions of the coset manifold. We see in particular that if $g^{-1} = g_1^{-1}g_2^{-1}g_1g_2$ is the commutator we then have,

$$g^{-1} \cdot y^a = y^a - \epsilon_1^A \epsilon_2^B [\mathbf{T}_A, \mathbf{T}_B]^a + \ldots \ . \tag{6.27}$$

Let us note the conventions used here. Given an operator $\Phi : M \to X$ we use **passive transformation** conventions,

$$U(g)^\dagger \Phi(y) U(g) = \Phi(y') \ . \tag{6.28}$$

so writing the operator $U(g) = e^{i\epsilon^A O_A}$ gives,

$$[O_A, \Phi] = -\mathbf{O}_A \Phi \ . \tag{6.29}$$

[18] A symmetric space is a space which for each point there is a symmetry fixing that point.

This $\mathbf{O}_A$ is a representation of the operator, a differential operator, acting on the space of operators. In particular, for the differential operators generated from elements of the Lie algebra $\mathfrak{g}$, the real vector fields are Killing vectors of the pseudo-Riemannian manifold M.

The $\mathcal{N}=1$ superspace in four dimensions. The four-dimensional $\mathcal{N}=1$ superspace can be defined as the coset,

$$\mathbb{R}^{1,3|4} \cong ISO(1,3|4)/SO(1,3) \ , \tag{6.30}$$

where we use the generators,

$$g = e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}+ix^\mu P_\mu+i\theta^\alpha Q_\alpha+i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} \in ISO(1,3|1) \ , \tag{6.31}$$

and label the cosets as,

$$\mathbf{x}(z) = e^{-ix^\mu P_\mu+i\theta Q+i\bar{\theta}\bar{Q}} \ . \tag{6.32}$$

The superspace coordinates are $y = (x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$. Here supersymmetry is realised geometrically as a translation along the fermionic coordinates of the superspace by the action of the group element,

$$g_F = e^{i\eta Q+i\bar{\eta}\bar{Q}} \ , \tag{6.33}$$

and in particular this gives the transformation of superspace coordinates using $g_F^{-1}\mathbf{x}(z)$ as,

$$x^\mu \mapsto x^\mu - i\eta\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\eta} \tag{6.34}$$
$$\theta \mapsto \theta + \eta \tag{6.35}$$
$$\bar{\theta} \mapsto \bar{\theta} + \bar{\eta} \tag{6.36}$$

This notation means the differential operator associated to the left-action of the group are the right-invariant vector fields,

$$\mathbf{Q}_\alpha = -i\left(\partial_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right) \tag{6.37}$$
$$\bar{\mathbf{Q}}_{\dot{\alpha}} = i\left(\partial_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right) \ , \tag{6.38}$$

which gives the **supercharges** as differential operators. Similarly, we can define **covariant derivatives** as the left-invariant vector fields acting on the cosets as a right action. The covariant derivatives will then have the coordinate description as follows,

$$\mathbf{D}_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{6.39}$$
$$\bar{\mathbf{D}}_{\dot{\alpha}} = \partial_{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \ . \tag{6.40}$$

Chiral superspace of 4d $\mathcal{N}=1$. Sometimes it is useful to redefine coordinates of the superspace such that the actions of the supercharge is manifest. In Question 2 of problem sheet 3 you have defined the chiral coordinates $(y^\mu, \vartheta^\alpha, \bar{\vartheta}_{\bar{\alpha}})$ [19] using the relation,

$$e^{-ix^\mu P_\mu+i\theta^\alpha Q_\alpha+i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} = e^{i\vartheta^\alpha Q_\alpha}e^{-iy^\mu P_\mu}e^{i\bar{\vartheta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} \ . \tag{6.41}$$

[19] Antichiral coordinates $(\hat{y}^\mu, \hat{\vartheta}^\alpha, \hat{\bar{\vartheta}}_{\bar{\alpha}})$ are simply defined the other way around.

Expanding using the BCS formula gives,

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{6.42}$$
$$\vartheta^\alpha = \theta^\alpha \tag{6.43}$$
$$\bar{\vartheta}_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} \ . \tag{6.44}$$

In these coordinates we will then have,

$$Q_\alpha = i\partial_\alpha \tag{6.45}$$
$$\bar{Q}^{\dot{\alpha}} = i\partial_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}} - 2(\vartheta\sigma^\mu)^{\dot{\alpha}}\partial_\mu \ . \tag{6.46}$$

Similarly the covariant derivatives are defined as the right action which gives

$$D_\alpha = -i\partial_\alpha + 2(\sigma^\mu\bar{\vartheta})_\alpha\partial_\mu \tag{6.47}$$
$$\bar{D}^{\dot{\alpha}} = -i\partial_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}} \ . \tag{6.48}$$

We see that in these coordinates the action of the supercharge Q_α is now simplified. There is no new physics here; only a mathematical redefinition of coordinates.

Grassmann variables. Perhaps it is important to discuss what the fermionic coordinates actually mean. We say the fermionic variables are *Grassmann-valued* if they satisfy the Grassmann algebra.

Definition 6.4. A **Grassmann algebra** is the exterior algebra $\bigwedge V$ of a vector space V over a field $\mathbb{K}$, equipped with an associative binary operation known as the **exterior product** $\wedge$.

In leyman terms, we drop the $\wedge$ and just write the Grassmann variables next to each other, and assign a negative sign whenever we swap fermionic variables θ_i. An exterior algebra can be therefore built up from variables θ_i which obey the relation,

$$\theta_i\theta_j = -\theta_j\theta_i \ , \tag{6.49}$$

and the algebra $\bigwedge V$ consists of all formal linear combinations of finite products of $\{\theta_i\}$. The exterior algebra naturally gives a $\mathbb{Z}_2$-grading split into linear combinations with even products and odd products:

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \ . \tag{6.50}$$

Now this gives another way of defining superspaces. We can define an (m, n)-dimensional superspace as,

$$\mathbb{R}^{m|n} = \Lambda_0 \times \cdots \times \Lambda_0 \times \Lambda_1 \times \cdots \times \Lambda_1 \ , \tag{6.51}$$

where there are m Λ_0s and n Λ_1s. An archetypical example is the 4d $\mathcal{N} = 1$ superspace — in this notation it will be $\mathbb{R}^{4|4}$, with coordinates (x^μ, θ^i) and θ^i are Majorana spinors. Typically we use Weyl spinors $\theta, \bar{\theta}$ so the fermionic coordinates are $(\theta^1, \theta^2, \bar{\theta}_{\dot{1}}, \bar{\theta}_{\dot{2}})$.

The integration rules over Grassmann variables are known as **Berezin integration** rules. For a Grassmann variable θ,

$$\int d\theta = 0 \ , \quad \int d\theta\theta = 1 \ . \tag{6.52}$$

So we can treat $\theta = \delta_0(\theta)$ as if it is a delta function. The measure is defined as $d^2\theta = \frac{1}{2}d\theta^1 d\theta^2$, so,

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta \ , \quad \int d^2\bar{\theta} = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}} \ . \tag{6.53}$$

There is a more sophisticated construction of superspace which involves constructing the space by understanding the functions of a space. In algebraic geometry this is known as a ringed space — where we define a manifold by understanding how the set (sheaf) of functions on the manifold behave. We will not discuss this in any detail here, the interested amongst you can look at Appendix E for details.

Superspace of other dimensions and $\mathcal{N}$. Before we continue let us ask the question - is the superspace formalism discussed above for four-dimensional $\mathcal{N} = 1$ supersymmetry generalisable to other dimensions and extended supersymmetry? There are a few points to note here:

1. In four dimensions there is no useful superspace formalism for $\mathcal{N} > 1$. The only useful one is for pure gauge $\mathcal{N} = 2$ theories.

2. In other dimensions, generally there is a superspace formalism for $N_Q \leq 4$ only in dimensions $d \leq 4$. This is because the general superfield (which we will define below again) of the form,

$$F(x, \theta) = f_0(x) + ... + \theta^1 \dots \theta^{N_Q} f_{N_Q}(x) \ , \tag{6.54}$$

   gives a total of $2^{N_Q - 1}$ components, much larger than the number of degrees of freedom expected in a SUSY irrep. There might not exist any consistent sets of constraints to give off-shell supermultiplets in general [18].

6.3 Superfields

Now that we have defined what a superspace is, we can properly define a superfield. For the remainder of this section we will always work with 4d $\mathcal{N} = 1$ superspace unless specified.

Definition 6.5. A **superfield** is a field of superspace, defined as,

$$\Phi : M^{m|n} \to X \ . \tag{6.55}$$

To put it simply, a superfield is basically just a function that depends on superspace coordinates. Let us contrast this with component fields. Component fields are ordinary fields on Minkowski space, namely,

$$\phi : M^{1,n-1} \to X \ , \tag{6.56}$$

where X is known as the **target space**. For fermionic fields we need to introduce some parity-reversal space (so it is Grassmann-valued) so we write,

$$\psi : M^{1,n-1} \to x^*\Pi TX \ . \tag{6.57}$$

The details of the notations will be explored in Appendix E. To go between component fields and superfields, we can always expand a superfield in terms of Grassmann coordinates:

$$\begin{aligned} Y(x,\theta,\bar\theta) = f(x) + \theta\psi(x) + \bar\theta\bar\psi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) \\ + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta d(x) \ . \end{aligned} \tag{6.58}$$

There is in fact a way to check the supercharges in differential operator form in Eq.(6.37). Recall that for a field the infinitesimal transform can be generated by the Lie algebra,

$$\phi(x+a) = e^{-iaP}\phi(x)e^{iaP} = \phi(x) - ia^\mu[P_\mu, \phi(x)] \ . \tag{6.59}$$

So we can compare the variation of a superfield along the fermionic coordinates,

$$Y(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\bar\theta) = e^{-i(\epsilon Q+\bar\epsilon\bar Q)}Y(x,\theta,\bar\theta)e^{i(\epsilon Q+\bar\epsilon\bar Q)} \ , \tag{6.60}$$

and using BCH to get,

$$\delta x^\mu = i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta \tag{6.61}$$

$$\delta\theta^\alpha = \epsilon^\alpha \ , \tag{6.62}$$

$$\delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha} \ , \tag{6.63}$$

we can expand,

$$\delta_{\epsilon,\bar\epsilon}Y = Y - i\epsilon^\alpha[Q_\alpha, Y] + i\bar\epsilon^{\dot\alpha}[\bar Q_{\dot\alpha}, Y] \ , \tag{6.64}$$

and we get the differential operators,

$$Q_\alpha = -i\left(\partial_\alpha - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\right) \tag{6.37}$$

$$\bar Q_{\dot\alpha} = i\left(\partial_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\right) \ , \tag{6.65}$$

exactly as before.

To recap — the superfields contain component fields — to get the component fields from a superfield we simply expand with respect to the fermionic coordinates to get the ordinary bosonic and fermionic fields.

6.4 Superfield formalism

I want actions. They are cool and give me theories. Now that we have superfields and component fields, the question is: How do I construct SUSY-invariant actions? In particular, how do I write down theories that involve the fields from one supermultiplet, such that supersymmetry is manifest in the theory? There are two ways of doing this

1. **Component formalism.** We could start with component fields of a supermultiplet, check the supersymmetric-variables and add relevant auxiliary component fields to capture off-shell degrees of freedom. This is what we have done with the chiral multiplet which allowed us to construct the Wess-Zumino action.

2. **Superfield formalism.** Alternatively, we can use superfields — we can start with superfields and integrate over superspace. We shall see that this automatically gives SUSY-invariant theories, whilst extra constraints must be imposed on a generic superfield Y as it has too many field components to be an irreducible representation of SUSY.

For 4d $\mathcal{N} = 1$ supersymmetric theories, the superfield formalism is the easiest way to build supersymmetric actions. The reason why is this — given a superfield Y, the action is,

$$S = \int d^4x d^2\theta d^2\bar{\theta} Y \ . \tag{6.66}$$

Let us consider varying this with respect to SUSY,

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x d^2\theta d^2\bar{\theta} Y = \int d^4x d^2\theta d^2\bar{\theta} \delta_{\epsilon,\bar{\epsilon}} Y \ , \tag{6.67}$$

but since,

$$\delta_{\epsilon,\bar{\epsilon}} Y = \epsilon^\alpha \partial_\alpha Y + \bar{\epsilon}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} Y + \partial_\mu [-i(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon})Y] \ , \tag{6.68}$$

and since the first two terms give zero under fermionic integration and the last term is a total derivative, we see that $S = \int d^4x d^2\theta d^2\bar{\theta} Y$ is automatically SUSY-invariant. This obviously applies to any product (or formal linear combinations) of superfields $\mathcal{A} = \mathcal{A}(Y)$. Integrating over the Grassmann variables give,

$$S = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{A} = \int d^4x \mathcal{L}(x) \ , \tag{6.69}$$

where $\mathcal{L}(x)$ is the Lagrangian in the normal sense.

Now we need to impose extra conditions such that the superfields furnish the field content of one superfield. The constraints we have seen in the lectures are:

1. **Chiral superfields.** Imposing the chiral constraint lead to chiral superfields. Given Φ a superfield, the chiral constraint is,

$$\bar{D}_{\dot{\alpha}} \Phi = 0 \ , \tag{6.70}$$

where D_α and $\bar{D}_{\dot{\alpha}}$ are the covariant derivatives defined in Eq.(6.39) and (6.40) respectively. It satisfies,

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \ . \tag{6.71}$$

Notice that the condition in Eq.(6.70) is not a real constraint. This produces a chiral superfield which furnishes the 4d massless chiral multiplet content, generated from the Clifford vacuum $|\Omega_0\rangle$.

The easiest way to analyse this is to go to chiral superspace with $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. In these coordinates the chiral superfield has component fields,

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y) \ . \tag{6.72}$$

The supersymmetry transformations can be checked with

$$\delta_{\epsilon,\bar{\epsilon}}\Phi = (i\epsilon Q + i\bar{\epsilon}\bar{Q})\Phi(y,\theta) \ , \tag{6.73}$$

effected by

$$Q_\alpha = -i\partial_\alpha \ , \tag{6.74}$$

$$\bar{Q}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \ . \tag{6.75}$$

This gives,

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\Phi &= (\epsilon^\alpha\partial_\alpha + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\beta}}\partial_\mu)\Phi(y,\theta) \\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta(-\sqrt{2}\epsilon F + \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_\mu\phi) - \theta\theta(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi) \ ,\end{aligned} \tag{6.76}$$

so we can extract $\delta\phi$, $\delta\psi_\alpha$ and δF directly from the expression. Here I have used the short-hand notation $\partial_\mu = \frac{\partial}{\partial y^\mu}$.

2. **Vector superfields.** Imposing the real constraint lead to vector superfields. For V a superfield the real constraint is,

$$V = \bar{V} \ . \tag{6.77}$$

This produces a vector superfield which furnishes the 4d massless vector multiplet content, generated from the Clifford vacuum $\left|\Omega_{\frac{1}{2}}\right\rangle$.

The general expansion is of the form,

$$\begin{aligned}V(x,\theta,\bar{\theta}) = &\, C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &+ \frac{i}{2}\theta\theta(M(x) + iN(x)) - \frac{i}{2}\bar{\theta}\bar{\theta}(M(x) - iN(x)) \\ &+ i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) - i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\ &+ \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) - \frac{1}{2}\partial^2 C(x)\right) \ .\end{aligned} \tag{6.78}$$

This superfield after gauge-fixing will have 4 bosonic and 4 fermionic off-shell degrees of freedom. Under the transformation,

$$V \mapsto V + (\Phi + \bar{\Phi}) \ , \tag{6.79}$$

the vector component transforms as,

$$v_\mu \mapsto v_\mu - \partial_\mu(2\,\mathrm{im}(\phi)) \ , \tag{6.80}$$

which is exactly how an ordinary abelian gauge transformation acts on a vector field. This is consistent with having a gauge field in the definition of a vector supermultiplet. The other components transforms as,

$$C \mapsto C + 2\text{Re}(\phi) \ , \tag{6.81}$$

$$\chi \mapsto \chi - i\sqrt{2}\psi \ , \tag{6.82}$$

$$M \mapsto M - 2\text{Im}(F) \ , \tag{6.83}$$

$$N \mapsto N + 2\text{Re}(F) \ , \tag{6.84}$$

$$D \mapsto D \ , \tag{6.85}$$

$$\lambda \mapsto \lambda \ , \tag{6.86}$$

so in light of this we can choose a gauge,

$$\text{Re}(\phi) = -\frac{C}{2} \ , \quad \psi = -\frac{i}{\sqrt{2}}\chi \ , \quad \text{Re}(F) = -\frac{N}{2} \ , \quad \text{Im}(F) = \frac{M}{2} \ . \tag{6.87}$$

This is known as the **Wess-Zumino gauge** where $C = M = N = \chi = 0$. Then the vector superfield has the following expansion,

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \ . \tag{6.88}$$

Notice that taking the Wess-Zumino gauge breaks supersymmetry — you will need a compensating transformation with Φ to restore supersymmetry.

3. **Linear superfields.** There are other types of superfields that accommodate composite operators – conserved currents and supercurrents. A **linear superfield** is a real superfield that satisfies the extra constraint,

$$D^2\mathcal{J} = \bar{D}^2\mathcal{J} = 0 \ , \tag{6.89}$$

with the component expression,

$$\mathcal{J} = J(x) + i\theta(x)j(x) - i\bar{\theta}\bar{\jmath}(x) + \theta\sigma^\mu\bar{\theta}j_\mu(x) + \frac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu j(x) - \frac{1}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\jmath}(x) + \frac{1}{4}\theta^2\bar{\theta}^2J(x) \ , \tag{6.90}$$

where from the on-shell constraint Eq.(6.89) above we see that the current $j_\mu(x)$ is conserved, $\partial^\mu j_\mu = 0$. $J(x)$ is defined up to Schwinger terms entering the current algebra, with $[Q_\alpha, j_\mu] = \mathcal{O}_{\alpha\mu}$.

Similarly the supersymmetry current $S_{\alpha\mu}$ associated to the conservation of the fermionic charge Q_α. The Schwinger terms can now be seen from the equation $\{\bar{Q}_{\dot{\alpha}}, S_{\alpha\nu}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}T_{\mu\nu} + \mathcal{O}_{\alpha\dot{\alpha}\nu}$, and hence the energy-momentum tensor and supercurrent sits in the same superfield. There are many different completions of the superfield, the most well-known being the **Ferrara-Zumino multiplet**, described by a pair of superfields $(\mathcal{J}_\mu, X)$ satisfying the relation,

$$2\bar{D}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{J}_\mu = D_\alpha X \ , \tag{6.91}$$

where $\mathcal{J}$ is a real vector superfield and X is a chiral superfield. An alternative superfield accomodating the energy-momentum tensor and the supercurrent is known as the **$\mathcal{R}$-multiplet**, which can be defined as a pair of superfields $(\mathcal{R}_\mu, \chi_\alpha)$ with the on-shell condition,

$$2\bar{D}^{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}\mathcal{R}_\mu = \chi_\alpha \ , \tag{6.92}$$

with $\mathcal{R}_\mu$ a real vector superfield and χ_α a superfield. The details of the superfield can be found in [19], for example.

Now we can think about what terms can contribute to a supersymmetric Lagrangian. Since the supersymmetric Lagrangian can be invariant up to a total derivative,

$$\delta\mathcal{L} = \partial_\mu(V^\mu) \ , \tag{6.93}$$

we see that there are two possible contributions to the **supersymmetric Lagrangian**.

1. **D-term.** For a general superfield, we can do a component expansion and check that the top component, i.e. the component associated to $\theta\theta\bar{\theta}\bar{\theta}$, is (upon a rescale of $D \mapsto D - \frac{1}{2}\partial^2 C$ with C the scalar component),

$$\partial D = -\epsilon\sigma^\mu\partial_\mu\bar{\lambda} + \bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\lambda = \partial_\mu\left(-\epsilon\sigma^\mu\bar{\lambda} + \bar{\epsilon}\bar{\sigma}^\mu\lambda\right) \ . \tag{6.94}$$

   The second equality follows from the fact that in global supersymmetry the parameter ϵ is a constant. So it is clear that a real superfield $\mathcal{S}$ will contribute to the Lagrangian as,

$$\mathcal{L}_D = \int d^2\theta d^2\bar{\theta}\mathcal{S} \ . \tag{6.95}$$

   This is known as the D-term contribution.

2. **F-term.** The other contribution exists when we have chiral superfields. Recall that the auxiliary field of a chiral superfield Φ transforms as,

$$\delta F = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \ . \tag{6.96}$$

   This is obviously a total derivative when ϵ is constant, so we get the contribution,

$$\mathcal{L}_F = \int d^2\theta\Phi \ , \quad \mathcal{L}_{\bar{F}} = \int d^2\theta\bar{\Phi} \ . \tag{6.97}$$

   Here we integrate half of superspace and this is known as the F-term contribution.

The above ingredients are sufficient for us to generate supersymmetric field theories.

7 Non-linear sigma models

Sigma models are crucial in quantum field theories. What we will do in this section is to give an overview on what sigma models are in a semi-formal manner, and understand how they will arise in the context of supersymmetric theories.

Let us begin with a definition for sigma models.

Definition 7.1. An n**-dimensional sigma model** is an n-dimensional quantum field theory which is encoded by geometric data. It describes physical configuration spaces that are mapping spaces into a geometric space equipped with some differential geometric structure.

A bit of a confusing definition. Let us clarify what a **mapping space**. Given two topological spaces X and Y, one can consider the space of maps $f : X \to Y$. This is the mapping space between X and Y, denoted by $\text{Map}(X, Y)$. Therefore, we can have the following alternative definition.

Definition 7.2. A **sigma model** is quantum field theory which is a mapping space (the space of maps) $\phi : \Sigma \to X$, where

- Σ, commonly known as the **configuration space**. This is the space of configuration, for example, in the case of a particle, there is the space of 'proper time'.
- X, commonly known as the **target space**. this is the geometric space that the field $\phi : \Sigma \to X$ takes value in. Typically this space will has some additional differential geometrical properties with it.

The elements in the mapping space $\phi \in \text{Map}(\Sigma, X)$ are called **fields**.

Sigma models can be further categorised into two different types.

Definition 7.3. A **linear sigma model** is a sigma model where the target space is isomorphic to some flat space, $X \cong \mathbb{R}^n$. A **non-linear sigma model** is a sigma model where the target space is not flat.

Let us look at some examples.

Example 7.1 (Classical σ-models)**.** The simplest example from classical field theory is the sigma model of a relativistic particle. In this case, the sigma model is defined by the following data.

- The target space X is a pseudo-Riemannian manifold (X, g), or spacetime.
- The parameter space $\Sigma = \mathbb{R}$ is a real line, the abstract worldline of the particle.
- The background gauge field is the one-form connection of the bundle $S^1 \to X$, $A \in \Omega^1(X)$. The curvature $F = dA$ is the field strength of an electromagnetic field on X.
- The configuration space is then the quotient,

$$C^\infty(\mathbb{R}, X)/\text{Diff}(\mathbb{R}) \, , \tag{7.1}$$

  with the covariant phase space the subspace of the space of configurations which satisfy the equations of motion. Each object in this configuration space is then a particle trajectory $\gamma : \Sigma \to X$, with each morphism $\gamma_1 \cong \gamma_2$ a gauge transformation.

Example 7.2 (Linear sigma models)**.** Linear sigma models are typically used in understanding symmetry breaking. Take the example where we have a scalar field $\phi : \Sigma \to M$ where $\Sigma = M^{1,3}$, the Minkowski space and $M = \mathbb{R}^n$. The Lagrangian is,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi) \cdot (\partial^\mu \phi) + \frac{1}{2}\mu^2(\phi \cdot \phi) - \frac{\lambda}{4}(\phi \cdot \phi)^2 \tag{7.2}$$

where $\cdot$ indicates the dot-product on $\mathbb{R}^n$. This theory is invariant under an $O(N)$-transformation,

$$\phi_i = R_{ij}\phi_j \tag{7.3}$$

with ϕ_i indicating the i-th argument of ϕ. When $\mu^2 > 0$ and $\lambda > 0$, the potential is minimised when

$$\langle \phi_0 \cdot \phi_0 \rangle = \frac{\mu^2}{\lambda} \ . \tag{7.4}$$

Note that this condition imposes no conditions on the direction of ϕ_0 and only its length. In particular, there is a gauge choice which we can use to fix the field ϕ_0 to be,

$$\langle \phi_0 \rangle = (0, \ldots, 0, v) \ , \quad v = \frac{\mu}{\sqrt{\lambda}} \ . \tag{7.5}$$

We can do expand the field around the equilibrium as,

$$\phi_k(x) = \Big(\pi^k(x), v + \sigma(x)\Big) \ , \quad k = 1, \ldots, N-1 \ , \tag{7.6}$$

and we get a Lagrangian,

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}(\partial_\mu \pi_k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 \\ &- \sqrt{\lambda}\mu\sigma^2 - \sqrt{\lambda}\mu(\pi_k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi_k)^2\sigma^2 - \frac{\lambda}{4}(\pi_k)^4 \ , \end{aligned} \tag{7.7}$$

which we can see we obtain a massive σ field and a set of $N-1$ massless fields. This is just the usual spontaneous symmetry breaking where the original $O(N)$ symmetry is broken into $O(N-1)$. In fact, the full $O(N)$ symmetry remains — it is simply that the effective theory around the minimum now has reduced symmetry. The reduced symmetry is the symmetry of the vacuum manifold which is parametrised by $\pi_k(x)$, and the massive σ field describes excitations along the radial direction. We will see this more next time when we discuss the Higgs mechanism.

7.1 Non-linear sigma models in supersymmetric theories

So why do we care so much about non-linear sigma models in supersymmetric theories? The reason lies in the kinetic term of supersymmetric theories.

Under construction

Working in progress — chiral models.

7.2 Kähler geometry

Under construction

Working in progress — geometrical details?

8 Gauge theories and SUSY gauge theories

Under construction

Working in progress...

9 Moduli and supersymmetry breaking

Moduli spaces are not unique to supersymmetric field theories, but they are a key part of supersymmetric field theories. In this section let us investigate the origins of moduli spaces in supersymmetric theories.

9.1 SUSY moduli

Let us begin with a definition.

Definition 9.1. A **vacuum state** is the lowest energy state in a quantum field theory. A **supersymmetric vacuum** is the lowest energy state in a supersymmetric quantum field theory.

The investigation of supersymmetric vacua, as it turns out, boils down to the investigation of the scalar potential. This is encapsulated in the following proposition.

Proposition 9.1. *The supersymmetric vacua are in one-to-one correspondence with the zeros of the scalar potential.*

Proof. There are two principles that will lead to the proof.

Firstly, a vacuum is a Lorentz-invariant state configuration. This means that all field derivatives and fields apart from scalars should vanish in a vacuum state (so that no particular directions will be picked out). Clearly, the only terms relevant in the Hamiltonian is the scalar potential.

Now we recall that in a supersymmetric vacuum, the relation $\{Q, Q^\dagger\} \sim P$ gives the conclusion,

$$\langle\Omega| P^0 |\Omega\rangle \sim \sum_\alpha \left(||Q_\alpha |\Omega\rangle ||^2 + ||Q^\dagger_\alpha |\Omega\rangle ||^2 \right) \geq 0 \,. \tag{9.1}$$

Therefore $E_0 = 0$ if and only if it is a supersymmetric state. Conversely supersymmetry is broken (perturbatively) iff the vacuum energy is positive. □

Let us now recall what the scalar potential looks like in a typical supersymmetric theory. A supersymmetric theory with gauge-matter interactions (with a vector and chiral multiplet) has the scalar potential,

$$V(\phi, \bar{\phi}) = \frac{\partial W}{\partial \phi^i} \frac{\partial \bar{W}}{\partial \bar{\phi}^i} + \frac{g^2}{2} \sum_a |\bar{\phi}^i (T^a)_{ij} \phi^j + \xi^a|^2 = \bar{F} F + \frac{1}{2} D^2 \geq 0 \,. \tag{9.2}$$

Here the F and D fields are the auxiliary fields for the chiral and vector multiplet separately. The argument for setting $V = 0$ means that the supersymmetric vacua are described by D-term and F-term equations.

$$\bar{F}_i(\phi) = 0 \, , D^a(\phi, \bar{\phi}) = 0 \, . \tag{9.3}$$

To find the moduli of supersymmetric vacua (which are the lightest degrees of freedom of the low-energy effective theory and hence the space of flat directions that the potential does not depend on), we make the following steps.

1. **D-flat directions.** We typically start with the space of scalar field VEVs such that

$$D^A(\phi, \bar{\phi}) = 0 \, . \tag{9.4}$$

   This term always exist when the supersymmetric model contains gauge-matter interactions.

2. **F-flat directions.** If a superpotential is present, F-term equations may put additional constraints on the D-flat directions. These are the equations,

$$\bar{F}^i(\phi) = 0 \, . \tag{9.5}$$

The two conditions together give the so-called **classical moduli space** and represents the space of classical supersymmetric vacua.

9.2 Supersymmetry breaking

There are two types of supersymmetry breaking.

1. **Spontaneous.** In this case the scalar potential allows for a SUSY-breaking vacua.

2. **Explicit.** We can also add to the Lagrangian terms that explicitly break supersymmetry. These terms typically enter in the IR as irrelevant operators so in the UV they are negligible.

> Under construction
>
> Working in progress... need diagrams for supersymmetry breaking and types of supersymmetry breaking

Spontaneous supersymmetry breaking. Let us first consider the possibility that SUSY is broken spontaneously. From the analysis of the scalar potential above we can clearly derive two different ways of breaking supersymmetry.

- **F-term breaking.** If there is a non-trivial superpotential, one can set the F-term equations to zero. For a superpotential of the renormalisable type it appears in the following general form,

$$W = a_i \phi^i + m_{ij} \phi^i \phi^j + g_{ijk} \phi^i \phi^j \phi^k \, , \tag{9.6}$$

this means that the F-term equations are,

$$\bar{F}_i(\phi) = a_i + m_{ij}\phi^j + g_{ijk}\phi^j\phi^k \ . \tag{9.7}$$

We see that a non-zero linear term $a_i \neq 0$ is necessary for $\bar{F}_i = 0$, otherwise the trivial vacuum $\langle \phi^i \rangle = 0$ will satisfy the F-term equations. This condition holds even with a non-trivial Kähler potential.

The example explored in the example sheets is the O'Raifearteigh Model, with the superpotential,

$$W = \frac{1}{2}hX\Phi_1^2 + m\Phi_1\Phi_2 - \mu^2 X \ . \tag{9.8}$$

You can explore about this model in deeper sense in [**?**].

- D-term breaking. This occurs when the D-term is zero. Recall the D-term appears as,

$$D^a = g^2\left(\bar{\phi}^i (T^a)_{ij}\phi^j + \xi^a\right) \ , \tag{9.9}$$

  The first term can be always set to zero using the gauge-invariance of scalar fields, which states that $(T^a)_{ij}\phi^j = 0$. A non-trivial D-term breaking therefore requires the existence of the Fayet-Iliopoulos terms, i.e. it only occurs when there is an abelian field in the system. In this case, the relative sign between the FI term and the term in the front, i.e. $\sum_i q_i |\phi^i|^2$ will play a role. If both terms are of the same sign then $V = 0$ is impossible and therefore supersymmetry is always broken. However, for the case where there are of different signs, it is possible to find a D-term vacua where $D^a = 0$. Supersymmetry in this case may be preserved.

In the case where supersymmetry is not broken but gauge symmetry is broken, this is the supersymmetric Higgs model, as explored in Q4 in Sheet 4.

10 R-symmetry

Under construction

Working in progress...

11 Supersymmetry breaking

Under construction

Working in progress...

12 Renormalisation in supersymmetric theories

Under construction

Working in progress...

13 Supergravity — a primer

Under construction

Working in progress...

A Example sheets feedback

A.1 Problem sheet 1

A.1.1 Question 1 - Poincaré symmetry

The main goal of this question is to find out how operators transform under the Poincaré group. There are some points to note:

(a) Some of you missed the fact that $M_{\mu\nu}$ is antisymmetric. Of course this just comes from the Lorentz algebra [20], but you should write,

$$\hat{M}_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \ , \tag{A.1}$$

and not,

$$\hat{M}_{\mu\nu} = -2i(x_\mu \partial_\nu) \ . \tag{A.2}$$

If you write the last question, then you will need to also contract the indices with some antisymmetric objects $\lambda^{\mu\nu}$. Otherwise it will be incorrect.

(b) We want to check

$$U(\Lambda_1, a_1)U(\Lambda_2, a_2) = U(\Lambda_3, a_3) \ , \tag{A.3}$$

where

$$\Lambda_3 = \Lambda_1\Lambda_2 \ , \quad a_3 = \Lambda_1 a_2 + a_1 \ . \tag{A.4}$$

The main point is that you need to make sure that the argument of the operator, i.e. $\mathcal{O}(x)$, also transforms appropriately. This means, you should take the active transformation carefully into account [21]. In particular note that,

$$U(\Lambda_3, a_3)^{-1}\mathcal{O}(x)^{\mathcal{A}}U(\Lambda_3, a_3) = L(\Lambda_3)^{\mathcal{A}}_{\mathcal{B}}\mathcal{O}^{\mathcal{B}}(\Lambda_3^{-1}x - \Lambda_3^{-1}a_3) \ , \tag{A.5}$$

you should check that the argument of $\mathcal{O}^{\mathcal{B}}$ matches with the one obtained by carrying out two transformations U_1 and U_2 back-to-back.

[20] See Andre's course on Groups and Representations.
[21] In fact, everyone missed it.

(c) This part is well done (apart from the arguments of the operators which I have already commented on). The main thing to note here is the extra term obtained,

$$[M_{\mu\nu}, \mathcal{O}^{\mathcal{A}}(x)]_{\text{new term}} = (-\mathcal{S}_{\mu\nu})^{\mathcal{A}}{}_{\mathcal{B}}\,\mathcal{O}^{\mathcal{B}}(x)\ , \tag{A.6}$$

where $\mathcal{S}_{\mu\nu}$ is a representation of the Lorentz algebra. The term exists because the field is now in a different representation space of the Poincaré group. You can think of the scalar field infinitesimal transformation as giving the relation between the field at $\Lambda^{-1}x$ versus the field at x. But since the field is now in a nontrivial representation there must be an extra part coming from 'representation space-contribution'.

A.1.2 Question 2 - Clifford algebra and Lorentz generators

This is just algebra. I don't really have much to say about this. If you struggle then opening any kindergarten QFT manual should save you. In particular, the last part requires carefully writing out the indices of $\lambda^{\rho}{}_{\sigma} = \frac{i}{2}\lambda^{\mu\nu}(S_{\mu\nu})^{\rho}{}_{\sigma}$, and you will need to figure out how to write out $(S_{\mu\nu})^{\rho}{}_{\sigma}$.

A.1.3 Question 3 - The homomorphism $SL(2,\mathbb{C}) \to SO(1,3)$

This question aims to build the homomorphism

$$\Psi : SL(2,\mathbb{C}) \to SO(1,3)\ . \tag{A.7}$$

This is sometimes known as the **spinor map**. Note that this is a two-to-one map as we will later find out that $\ker\Psi = \{\pm\mathbb{1}_2\}$. The image of the map is the identity component of $SO_{\mathbb{R}}(1,3)$, denoted typically as $SO^{+}_{1,3}(\mathbb{R})$. We then have,

$$PSL_2\mathbb{C} = \frac{SL_2\mathbb{C}}{\mathbb{Z}_2} \cong SO^{+}_{1,3}(\mathbb{R})\ , \tag{A.8}$$

where we write $SL(2,\mathbb{C})$ as $SL_2\mathbb{C}$ and $SO(1,3)$ as $SO_{1,3}(\mathbb{R})$. Of course, since Ψ is a smooth map and that $SL(2,\mathbb{C})$ is simply-disconnected, the map will always land on the identity component of $SO(1,3)$, i.e. $SO^{+}(1,3)$ or $\Lambda^{\uparrow}_{+}$ [22]. This is why there seems to be an ambiguity when I define the map Ψ.

Anyway, moving on to the question. Some points to note.

- Parts (a) and (b) are generally well done. I also discussed these briefly in the class.
- Part (c) is all about spinor algebras. In particular we want to show,

$$\Lambda(A_1)^{\mu}{}_{\nu}\Lambda(A_w)^{\nu}{}_{\rho} = \Lambda(A_1A_2)^{\mu}{}_{\rho}\ , \tag{A.9}$$

$$\Lambda(A)^{\mu}{}_{\nu}\Lambda(A)^{\rho}{}_{\sigma}\eta_{\mu\rho} = \eta_{\nu\sigma}\ . \tag{A.10}$$

You should prove that,

$$\operatorname{Tr}\left(A_1^{\dagger}\bar{\sigma}^{\mu}A_1\sigma_{\nu}\right)\operatorname{Tr}\left(A_2^{\dagger}\bar{\sigma}^{\nu}A_2\sigma_{\rho}\right) \overset{!}{=} -2\operatorname{Tr}\left(A_2^{\dagger}A_1^{\dagger}\bar{\sigma}^{\mu}A_1A_2\sigma_{\rho}\right) \tag{A.11}$$

[22] This notation is typically used to indicate the proper orthochronous Lorentz group.

for the first result and similarly for the second. This requires a bit of spinor algebra which we have developed in Q4 and in the class, and you should use prove,

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}(\bar{\sigma}_{\mu})^{\dot{\gamma}\delta} = -2\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\delta} \ . \tag{A.12}$$

Don't try and cheat your way through — it's good practice.

- Part (d) is covered in the class — you should use the fact that since $A \in SL(2,\mathbb{C})$, this gives constraints on the matrix components. Most of you stated that $A = \pm\mathbb{1}_2$ implies $\Lambda^{\mu}{}_{\nu} = \eta^{\mu}{}_{\nu}$ which is saying that $\mathbb{Z}_2 \subset \ker\Psi$ but not the other way around. You need to show that $\pm\mathbb{1}_2$ are the only solutions to the kernel to complete the full argument by, for example, using the method mentioned in the class (i.e. set $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and imposing conditions).

- Parts (e) and (f) involve deriving infinitesimla versions of the map Ψ, i.e. the Lie algebra homomorphism,

$$\tilde{\Psi} : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{so}_{1,3}\mathbb{C} \ . \tag{A.13}$$

The key point is to first derive the infinitesimal version of $\Lambda(A)^{\mu}{}_{\nu}$,

$$\lambda^{\mu}{}_{\nu} = -\frac{1}{2}\,\mathrm{Tr}\left(\delta A^{\dagger}\bar{\sigma}^{\mu}\sigma_{\nu}\right) - \frac{1}{2}\,\mathrm{Tr}\left(\bar{\sigma}^{\mu}\delta A\sigma_{\nu}\right) \tag{A.14}$$

and note that this is antisymmetric [23]. Now try and prove,

$$\mathrm{Tr}\left(\delta A\sigma^{\mu\nu}\right)^{*} = -\,\mathrm{Tr}\left(\delta A^{\dagger}\bar{\sigma}^{\mu\nu}\right) \ , \tag{A.15}$$

which results in

$$\lambda_{\mu\nu} = 2\,\mathrm{Re}\,\mathrm{Tr}\left(\delta A\sigma_{\mu\nu}\right) \ . \tag{A.16}$$

The reverse of the map can be constructed by writing,

$$\delta A = y^{\mu\nu}\sigma_{\mu\nu} \ , \tag{A.17}$$

and try to evaluate $\mathrm{Tr}\left(\sigma^{\mu\nu}\sigma^{\rho\sigma}\right)$ to get $\lambda_{\mu\nu} = -2y_{\mu\nu}$.

We actually haven't computed the reverse map of Ψ. This is in fact,

$$A = e^{i\phi}\frac{\sigma_{\mu}\Lambda^{\mu}{}_{\nu}\bar{\sigma}^{\nu}}{2\sqrt{\Lambda^{\mu}{}_{\mu}}} \tag{A.18}$$

with $\mathrm{tr}\,A = e^{i\phi}|\,\mathrm{tr}\,A|$. The phase $e^{i\phi}$ can be determined up to ±1 by imposing $\det A = 1$ [24].

[23] You should check this as an exercise.
[24] For the details of this construction see Hugh Osborne's Group Theory notes, §4.3.

A.1.4 Question 4 - Spinor algebra

This question is the most important one this sheet. You shoudl be really comfortable with the spinor algebra manipulations. The important points are the following:

1. Undotted sum goes downwards from left to right, $\psi^\alpha \chi_\alpha$.
2. Dotted sum goes upwards from left to right, $\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$.
3. $\epsilon^{\alpha\beta} = \epsilon^{-\beta\alpha} = \epsilon_{\beta\alpha}$.
4. $\left(\sigma^\mu_{\alpha\dot{\alpha}}\right)^T = \sigma^\mu_{\dot{\alpha}\alpha}$.
5. $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon^{\beta\delta} (\sigma^\mu)_{\delta\dot{\gamma}}$.
6. $(\sigma^\mu)_{\alpha\dot{\beta}} (\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = -2\delta^\delta_\alpha \delta^{\dot{\gamma}}_{\dot{\beta}}$.

You should prove all of this, and then evaluate the identities in the question again (see [20] for some identities). In particular, for (c)(i), the Schouten identity is useful:

$$\epsilon_{\alpha\beta}\delta^\mu_\gamma + \epsilon_{\beta\gamma}\delta^\mu_\alpha + \epsilon_{\gamma\alpha}\delta^\mu_\beta = 0 \tag{A.19}$$

A.2 Problem sheet 2

A.2.1 Question 1 - Super Jacobi Identities

The main point of this question is to illustrate how to use spinor identities we have developed in the lectures and the last class to evaluate expressions. The main point to highlight here is that in evaluating the super Jacobi identities you should use the following lemma.

Lemma A.1. *The following identity holds.*

$$(\sigma_{\mu\nu})_\alpha{}^\gamma \epsilon_{\gamma\beta} + (\sigma_{\mu\nu})_\beta{}^\gamma \epsilon_{\alpha\gamma} = 0 \, . \tag{A.20}$$

Proof. This should be quite straightforward. First you should use the definition of $\sigma_{\mu\nu}$,

$$\sigma_{\mu\nu} = \frac{i}{4} \left(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu \right) \, . \tag{A.21}$$

Now we use the following identities,

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = -\epsilon^{\alpha\beta} (\sigma^\mu)_{\beta\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} \tag{A.22}$$

$$\epsilon^{\alpha\beta} = \epsilon_{\beta\alpha} \tag{A.23}$$

to show that Eq. (A.20) holds true. See Proposition 1.39 in [20] for details. □

This is a non-trivial result that you should explicitly prove in your attempt! Otherwise I will treat the attempt to be invalid.

A.2.2 Question 2 - Superconformal algebra

In this question we revisit the superconformal algebra and compute some commutation relations of the superconformal algebra. The main point of this question is really just to get you to be comfortable with manipulating (super-)conformal algebraic expressions. There are in fact two main points I would like to cover in more detail.

Dilatation generator and Scaling dimensions

In part (a) most of you struggled to find the commutation of the dilatation generator D with the Poincaré and superconformal supercharges Q^I_α and S^I_α, namely,

$$[D, Q^I_\alpha] = \frac{i}{2} Q^I_\alpha \, , \tag{A.24}$$

$$[D, S^I_\alpha] = -\frac{i}{2} S^I_\alpha \, . \tag{A.25}$$

The key point here is to realise that the only non-trivial identity is the super-Jacobi identity so we will use that to derive the expressions. You should be able to realise that from dimensional considerations and matching the spinor indices (Lorentz representations) on both sides that we must have,

$$[D, Q^I_\alpha] = i\lambda^{IJ} Q^J_\alpha \, , \tag{A.26}$$

where $\lambda^I \in \mathbb{C}$ a priori. Conjugating this gives,

$$[D, \bar{Q}^I_\alpha] = -i\bar{\lambda}^{IJ} Q^J_\alpha \, . \tag{A.27}$$

Now you can use the Jacobi identity - evaluating on both sides will eventually allow you to conclude that $\lambda^{IJ} = \frac{1}{2}\delta^{IJ}$ with no imaginary part [25]. This is the same with the S^I_α with just the opposite sign.

Central Charge

In part (b) you are asked to show that the central charges in the superconformal algebra must vanish. The key idea here is to use the fact that the central charge is the generator of the central extension of the algebra and therefore commutes with all elements,

$$[X, Z] = 0, \quad \forall X \in \mathcal{L} \, . \tag{A.28}$$

Hence we must have $[D, Z] = 0$. I will leave you with using the Jacobi identity to further show that $Z = 0$ identically in the superconformal case.

The rest of the question is just algebraic manipulations which I don't really have anything more to say.

A.2.3 Question 3 - Massless supermultiplets

Massless multiplets are important in constructing the low-energy spectrum of supersymmetric theories. There are three main things to note.

[25] In fact, the eigenvalues of the dilatation operator gives you the scaling dimensions which are real.

Degeneracy of states

Recall that we generate the massless supermultiplet by starting with the Clifford vacuum [26] $|\Omega_\lambda\rangle$ with helicity λ. We then repeatedly hit the Clifford vacuum with raising operators to generate the entire multiplet. The key point of this question is to note that the degeneracy of the states of helicity $\lambda + \frac{1}{2}n$, where n is the number of fermionic generators acted on the Clifford vacuum, is $\binom{\mathcal{N}}{n}$ for $\mathcal{N}$-extended supersymmetry. Many of you have loosely argued that $\sum_{n \text{ even}} \binom{\mathcal{N}}{n} = \frac{1}{2}\sum_n \binom{\mathcal{N}}{n}$ without a consistent argument. The quickest way, instead, is to realise that,

$$0 = \sum_{n=0}^{\mathcal{N}} \binom{\mathcal{N}}{n} (-1)^n = (1 + (-1))^{\mathcal{N}} , \tag{A.29}$$

using the binomial theorem, and realising that $(-1)^n$ is indeed the eigenvalue of the fermionic number operator $(-1)^F$ acted on each level (up to a sign) so the above expression effectively captures the index $n_B - n_F$.

Multiplets as representations of $U(\mathcal{N})$

The next point to note is how the multiplet content at each level forms a representation of the maximal R-symmetry of the extended supersymmetry algebra. First recall in four-dimensions the maximal R-symmetry group of the $\mathcal{N}$-extended SUSY is $U(\mathcal{N})$ (a priori). The particles on each level of the multiplet then furnish the antisymmetric part of the Clebsch-Gordan decomposition of products of fundamental representations, as they are generated by fermionic operators which algebraic structure is isomorphic to the exterior algebra [27]. In particular, we can use tensor notations to indicate the particles in a multiplet [28]. You can read §**??** for a more detailed discussion.

There is a subtlety however. The 4d R-symmetry group is not always the full $U(\mathcal{N})$ group. In the case $\mathcal{N} = 2$, the R-symmetry group is in fact,

$$U(2)_R \cong U(1)_R \times SU(2)_R , \tag{A.30}$$

and for $\mathcal{N} = 4$, the R-symmetry group is $SU(4)$. The reason for the latter is because the fermionic fields actually realise the spinor representation of Spin(6) and Spin(6) $\cong SU(4)$. The fact the $U(1)$ subgroup of $U(\mathcal{N})$ R-symmetry is sometimes not well-explained - under CPT-conjugation, the multiplet maybe self-conjugate and therefore the $U(1)$ part of the full R-symmetry group coincides with the helicity group ($\mathfrak{u}(1) \cong \mathfrak{so}(1,1)$) and therefore plays no important role. This is, for example, realised for $\mathcal{N} = 4, 8$ cases, so the R-symmetry group is sometimes instead listed as $SU(4)$ and $SU(8)$ respectively.

CPT-completion of 4d $\mathcal{N} = 3$ vector multiplet

[26] Note that the Clifford vacuum here is not a vacuum in the usual QFT-sense. The Clifford vacuum is only the **lowest weight state** in the super-Poincaré algbera, and is not necessarily the state with minimal energy.

[27] Using Young diagram notation, the particles will be represented by irreps corresponding to columns of boxes. The number of boxes corresponds to the number of raising operators acted on the Clifford vacuum (i.e. level in the multiplet).

[28] You should have learnt this in Groups and Representations, see Andre's lectures notes.

The second part of the question asks you to compute the four-dimensional $\mathcal{N}=3$ multiplet with the Clifford vacuum $|\Omega_{-1}\rangle$. Now if you start with the Clifford vacuum $|\Omega_{-1/2}\rangle$ instead, you will get the CPT-conjugate of the first multiplet. Therefore after CPT-completion the $\mathcal{N}=3$ multiplet doubles in size, and you can check as a part of the question that it exactly matches the $\mathcal{N}=4$ hypermultiplet (generated from $|\Omega_{-1}\rangle$). Therefore by convention this CPT-completed supermultiplet is referred to as the $\mathcal{N}=4$ multiplet and the $\mathcal{N}=3$ multiplets are forgotten. The decomposition actually occurs in general for higher extended supersymmetric multiplets — in general they can be expressed as compositions of lower multiplets. So as far as non-gravitational theories are discussed, the $\mathcal{N}=3$ multiplets are typically neglected.

A.2.4 Question 4 - Massive supermultiplets

I don't really have a lot to say about this question — this is just standard irrep constructions that you should be able to find in most supersymmetry textbooks. The only thing I would comment on would be how to write down the states at the second-level. Using the notation

$$a_\alpha^{(k)} = \frac{1}{\sqrt{2m}} Q_\alpha^{(k)} \ , \quad a_\alpha^{(k)\dagger} = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{\alpha}}^{(k)} \ , \tag{A.31}$$

we can generate the particles at the second-level by acting two fermionic operators. In particular, since we have defined the $\alpha = 1$ operator to raise the spin by a half, spin-0 states at level two should have the form

$$\epsilon^{\alpha\beta} \left(a_\beta^{(k)}\right)^\dagger \left(a_\alpha^{(l)}\right)^\dagger |\Omega_0\rangle \ , \tag{A.32}$$

with the epsilon ensuring that the particle generated is indeed an irrep. On the other hand the spin-1 particles at the same level have the form

$$\mathcal{S}_{kl} \left[\left(a_2^{(k)}\right)^\dagger \left(a_1^{(l)}\right)^\dagger\right] |\Omega_0\rangle \ , \tag{A.33}$$

with $\mathcal{S}_{kl}$ the symmetric operator acting on the indices k and l. Most of you didn't explicitly write this and I would imagine if you are forced to write this in an exam you would miss the (anti-)symmetrising factors so I have included them here.

The main discussion for constructing massive multiplets with central charges is now covered in detail in §5. I would advise studying this in detail for the exam.

A.3 Problem sheet 3

This sheet mainly concerns the coset formulation of superspace as well as supersymmetric chiral Lagrangians. The sheet involves many algebraic computations, but I do want to highlight some general points listed below.

A.3.1 Question 1 - Lorentz generators in superspace

We already know that Lorentz generators can be naturally represented in the superspace formalism. The main thing to note is that under the coset formalism as discussed in the lectures, the quotient space G/H with G super-Poincaré and H the Lorentz subgroup the representative is,

$$g_c(x,\theta,\bar{\theta}) = \exp\left(-ix^\mu P_\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right) \ . \tag{A.34}$$

A general element of the super-Poincaré group g induces a motion $(x,\theta,\bar{\theta}) \mapsto (x',\theta',\bar{\theta}')$ in superspace by,

$$g \cdot g_c(x,\theta,\bar{\theta}) = g_c(x',\theta',\bar{\theta}')h(x,\theta,\bar{\theta};g) \ . \tag{A.35}$$

Many of you have successfully evaluated the left-hand side to first order using the BCH formula, but have neglected the Lorentz transformation h as,

$$h(x,\theta,\bar{\theta};g) = \exp\left(\frac{i}{2}\tilde{\lambda}^{\mu\nu}M_{\mu\nu}\right) \ . \tag{A.36}$$

Therefore, we should expand both sides of Eq. (A.34) (expanding all terms, including the ones involving $M_{\mu\nu}$) to get the following four relations,

$$\lambda^{\mu\nu} = \tilde{\lambda}^{\mu\nu} \tag{A.37}$$

$$x^\mu - \frac{1}{2}\lambda^{\mu\nu}x_\nu = x'^\mu - \frac{1}{2}\tilde{\lambda}^{\mu\nu}x'_\nu \tag{A.38}$$

$$\theta^\alpha - \frac{1}{4}\lambda^{\mu\nu}\theta^\beta(\sigma_{\mu\nu})_\beta{}^\alpha = \theta'^\alpha + \frac{1}{4}\tilde{\lambda}^{\mu\nu}\theta'^\beta(\sigma_{\mu\nu})_\beta{}^\alpha \tag{A.39}$$

$$\bar{\theta}_{\dot{\alpha}} - \frac{1}{4}\lambda^{\mu\nu}\bar{\theta}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} = \bar{\theta}'_{\dot{\alpha}} + \frac{1}{4}\tilde{\lambda}^{\mu\nu}\bar{\theta}'_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \tag{A.40}$$

which should eventually give you the result,

$$\delta x^\mu = -\lambda^{\mu\nu}x_\nu \tag{A.41}$$

$$\delta\theta^\alpha = -\frac{1}{2}\lambda^{\mu\nu}\theta^\beta(\sigma_{\mu\nu})_\beta{}^\alpha \tag{A.42}$$

$$\delta\bar{\theta}_{\dot{\alpha}} = -\frac{1}{2}\lambda^{\mu\nu}\bar{\theta}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \tag{A.43}$$

and the differential operator defined by

$$\delta x^\mu\frac{\partial}{\partial x^\mu} + \delta\theta^\alpha\frac{\partial}{\partial\theta^\alpha} + \delta\bar{\theta}_{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \ . \tag{A.44}$$

A.3.2 Question 2 - Chiral superspace

A very neat way of defining the chiral coordinates in superspace is by the relations,

$$e^{-ix^\mu P_\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} = e^{i\vartheta^\alpha Q_\alpha}e^{-iy^\mu P_\mu}e^{i\bar{\vartheta}_{\bar{\alpha}}\bar{Q}^{\dot{\alpha}}} \ . \tag{A.45}$$

Many of you didn't use the BCH formula correctly. Some of you have also confused between the non-chiral coordinates x^μ and the chiral coordinates y^μ. Here you should expand to linear-order in the operators and take into account the commutation relations of the algebra, which gives the RHS of Eq. (A.45) to be

$$e^{i\vartheta^\alpha Q_\alpha - iy^\mu P_\mu + i\bar{\vartheta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} + \frac{1}{2}\left[i\vartheta^\alpha Q_\alpha, i\bar{\vartheta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right]} \ . \tag{A.46}$$

which should give you,

$$y^\mu = x^\mu + i\vartheta\sigma^\mu\,\delta\vartheta \tag{A.47}$$

$$\vartheta^\alpha = \theta^\alpha \tag{A.48}$$

$$\bar{\vartheta}_{\dot\alpha} = \bar{\theta}_{\dot\alpha}\ . \tag{A.49}$$

The anti-chiral coordinates are exactly defined analogously, with only a minus sign difference in the $\hat{y}^\mu$ definition. Following this kind of careful manipulation, you should be able to deduce the other parts of the question correctly. In particular, we see that the left- and right-actions of the operator,

$$Z = e^{i\eta^\alpha Q_\alpha + i\bar{\eta}_{\dot\alpha}\bar{Q}^{\dot\alpha}} \tag{A.50}$$

exactly generates the differential operators,

$$i\eta^\alpha Q_\alpha + i\bar{\eta}_{\dot\alpha}\bar{Q}^{\dot\alpha} \tag{A.51}$$

and

$$i\eta^\alpha D_\alpha + i\bar{\eta}_{\dot\alpha}\bar{D}^{\dot\alpha} \tag{A.52}$$

respectively with the operators defined as,

$$Q_\alpha = -i\frac{\partial}{\partial\vartheta^\alpha} \tag{A.53}$$

$$\bar{Q}_{\dot\alpha} = -i\left[-\frac{\partial}{\partial\bar{\vartheta}^{\dot\alpha}} + 2i(\vartheta\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial y^\mu}\right] \tag{A.54}$$

$$D_\alpha = -i\left[\frac{\partial}{\partial\vartheta^\alpha} + 2i(\sigma^\mu\bar{\vartheta})_\alpha\frac{\partial}{\partial y^\mu}\right] \tag{A.55}$$

$$\bar{D}_{\dot\alpha} = i\frac{\partial}{\partial\bar{\vartheta}^{\dot\alpha}}\ . \tag{A.56}$$

The reason why such a definition is possible is because the fields Q_α and D_α are defined as right-invariant and left-invariant vector fields in superspace. To see why right-actions lead to left-invariant vector fields, recall that in elementary Lie theory that a smooth right action of a Lie group G on a smooth manifold M of the form,

$$\theta : M\times G \to M, \quad (p,g)\mapsto p\cdot g\ , \tag{A.57}$$

can be generated by a one-parameter subgroup of $X\in \mathrm{Lie}(G)$,

$$(t,p)\mapsto p\cdot e^{tX}\ . \tag{A.58}$$

The infinitesimal generator of this flow $\hat{X}\in\mathfrak{X}(M)$ is,

$$\hat{X}_p = \left.\frac{d}{dt}\right|_{t=0} p\cdot e^{tX}\ . \tag{A.59}$$

The vector fields X are left-invariant, which ensures that the generator above is well-defined,

$$\left.\frac{d}{dt}\right|_{t=0}(p\cdot g)\cdot e^{tX_{g'}} = \left.\frac{d}{dt}\right|_{t=0} p\cdot e^{tX_{gg'}}\ . \tag{A.60}$$

The other notion is similar, see for example [17] for details.

The last part simply calls for the verification of the definitions of Q_α, $\overline{Q}_{\dot\alpha}$, D_α and $\overline{D}_{\dot\alpha}$ in terms of non-chiral coordinates which just requires the careful application of chain rule.

A.3.3 Question 3 - Chiral superfields

I don't really have much to say about this question - this question simply involves computing the SUSY variations of the chiral superfield and its component fields in the superspace formalism, which is standard in the literature. For details you should see for example, [20**?** , 21].

A.3.4 Question 4 - SUSY invariant actions

The algebraic manipulations in this question may be tricky but the physical insight is perhaps more interesting. Firstly, part (a) of the question is again standard manipulations of spinor identities that you should verify on your own - most of you just simply refused to calculate the terms properly. The spinor identities in Q4 of the first sheet should be prove to be helpful. For the last two parts, it may be helpful to follow the guidance sketched out in the sheet and discard any total derivatives that appear in intermediate steps. The end result in the SUSY current of the form,

$$J^{\mu}_{\alpha} = \sqrt{2}(\sigma^{\nu}\bar{\sigma}^{\mu}\psi)_{\alpha}\partial_{\nu}\bar{\phi} + -i\sqrt{2}\frac{\partial \bar{W}}{\partial \bar{\phi}}_{\alpha} . \tag{A.61}$$

The zeroth-component of this SUSY current is of course, according to the Noether procedure, the supercharge. This is in fact the starting point for analysing theories with local supersymmetries where extra terms are added to compensate the terms with the supercurrent - also known as **supergravity**.

A.4 Problem sheet 4

This sheet involves a lot of different applications of supersymmetric concepts in different contexts.

A.4.1 Question 1 - R-symmetry in superspace

This question mainly deals with R-symmetries in supersymmetry. The first part of the question is extremely straightforward - it should be straightforward to do an expansion of coordinates using the superfield and derive the transformation properties of the component fields. To argue that the superfield $D_\alpha \mathcal{S}$ has a definite R-character $R[\mathcal{S}] - 1$, you need to show explicitly using chain rule that both terms indeed has a factor of e^{-t} that factors out. The conjugate field $\bar{\mathcal{S}}(x, \theta, \bar{\theta})$ should have its arguments unchanged under conjugation, i.e.

$$\mathcal{S}^{\dagger}(x, e^{-it}\theta, e^{it}\bar{\theta}) = \overline{\mathcal{S}}(x, e^{-it}\theta, e^{it}\bar{\theta}) , \tag{A.62}$$

which you should explicitly check.

The most straightforward way to deduce the R-charge of $d^2\theta$ is to use the properties of the Berezin integration that differentiation acts in the same way as integration to deduce that,

$$R[d^2\theta] = -2 , \tag{A.63}$$

since $\theta^\alpha \mapsto e^{it}\theta$ so differentiation gives a factor e^{-it} by taking out (relatively) a θ from the expression. The more elegant way is to look at the Jacobian,

$$d^2\theta \mapsto d^2\theta \left[\det \frac{\partial \theta'^\alpha}{\partial \theta^\beta}\right] \tag{A.64}$$

with $\theta'^\alpha = e^{it}\theta^\alpha$. You should check that this gives the same factor as our very slick argument. Finally, the D-term and F-term transformations should be straightforwardly deduced by demanding the invariance of the action under R-symmetry.

A.4.2 Question 2 - Non-abelian vector superfields

I really don't have much to say for this question — this is just straightforward algebra manipulation. You should check that all the expressions add up to give you the final expression.

A trick however is to realise that you can neglect the $e^{t\,\mathrm{ad}_B}$ inside the generating function as to linear order in B this is not important.

A.4.3 Question 3 - Supersymmetric vacua in Wess-Zumino models

This question involves finding supersymmetric vacua given a chiral model with a certain superpotential. As explained in §, for chiral supersymmetric models we look at the F-term contributions to the potential which appears in the form,

$$V = F_i \bar{F}_i, \quad F_i = \frac{\partial W}{\partial \phi_i}\,, \tag{A.65}$$

with ϕ_i the bosonic chiral superfield in the chiral multiplet Φ_i. This question illustrates the many possibilities of the vacuum space:

1. Loci of algebraic equations (distinct vacua or a continuous vacuum space).
2. No supersymmetric vacua obtained.

Notice that such a formalism only allows the deduction of the existence of the classical supersymmetric vacuum space. To determine whether vacua exists, you will need to compute the minima of the potential (as in non-supersymmetric theories).

A.4.4 Question 4 - Supersymmetric Higgs mechanism in SQED

The key idea of this question is discussed in the class — you can find a summary of the Higgs and super-Higgs mechanism in §. The main problem was only silly algebraic mistakes.

B Localisation

In this section we sketch out two properties of supersymmetric Lagrangians in zero dimensions.

- **Localisation.** Partition function localises around critical points of the superpotential in a supersymmetric theory.

- **Deformation Invariance.** Partition function is invariant under the change in the potential.

We will sketch out these two ideas in more detail. We will see how supersymmetric QFTs have a special property where the partition function localises to specific points in the functional space and how this is related to topological quantities. This section is mainly based on David Skinner's SUSY notes [29] and [22].

B.1 Localisation

The idea of localisation is simple - in a supersymmetric theory, the value of the relevant path integral reduces to a much smaller-dimensional integral. In some cases this reduces to counting contributions of certain points in the field space.

Let us illustrate this in the zero-dimensional case. Using Berezin integration rules, where

$$\int d\psi = 0 \, , \quad \int \psi d\psi = 1 \, , \tag{B.1}$$

the simplest form of a non-trivial action is of the form [30],

$$S(X, \psi_1, \psi_2) = S_0(X) - \psi^1 \psi^2 S_1(X) \, . \tag{B.2}$$

The partition function Z, for which we define as,

$$Z = \int \prod_i dX^i \prod_a d\psi^a e^{-S(X,\psi)} \, , \tag{B.3}$$

is then,

$$Z = \int dX e^{-S_0} S_1(X) \, , \tag{B.4}$$

using the Berezin integration rules. What is the simplest case for a supersymmetric transformation to exist? We can define a real function called the **superpotential** [31] $W : \mathcal{F} \to \mathbb{R}$ where $\mathcal{F}$ is the space of functions $x = X$ and defining, $\psi = \psi^1 + i\psi^2$ and $\bar{\psi}$ to be the conjugate variable let us write,

$$S_0(x) = \frac{1}{2}(\partial W(x))^2 \, , \tag{B.5}$$

$$S_1(x) = \partial^2 W(x) \, , \tag{B.6}$$

so

$$S(X, \psi, \bar{\psi}) = \frac{1}{2}(\partial W)^2 - \psi\bar{\psi}\partial^2 W \, . \tag{B.7}$$

Then the action S is invariant under the flow generated by the fermionic vector fields,

$$\mathcal{Q} = \psi \frac{\partial}{\partial x} + \partial W(x) \frac{\partial}{\partial \bar{\psi}} \, , \tag{B.8}$$

[29] The notes by David Skinner in Cambridge is where I have learnt a lot of mathematical physics from - they are really good and it would be a shame if you give them a miss!

[30] We will need at least two fermionic variables as the action is in the even algebra and $\psi^2 = 0$.

[31] We call this the superpotential for nomenclature reasons - this will become clear when we look at other theories.

$$\mathcal{Q}^\dagger = \psi\frac{\partial}{\partial x} - \partial W(x)\frac{\partial}{\partial\bar{\psi}} \ , \tag{B.9}$$

with the nontrivial transformations being,

$$\mathcal{Q}(x) = \psi \ , \tag{B.10}$$

$$\mathcal{Q}(\bar{\psi}) = \partial W(x) \ , \tag{B.11}$$

and similarly for $\mathcal{Q}^\dagger$ [32]. These vector fields are exactly odd derivations of $C^\infty(\mathbb{R}^{1|2})$ and are the supercharges that generate supersymmetries of this zero-dimensional theory. Looking at the anticommutators, we see that,

$$\{\mathcal{Q},\mathcal{Q}\} = 2\partial W(x)\psi\frac{\partial}{\partial\bar{\psi}} \ , \quad \{\mathcal{Q}^\dagger,\mathcal{Q}^\dagger\} = -2\partial W(x)\bar{\psi}\frac{\partial}{\partial\psi} \ , \tag{B.12}$$

$$\{\mathcal{Q},\mathcal{Q}^\dagger\} = -\partial W(x)\left(\psi\frac{\partial}{\partial\psi} - \bar{\psi}\frac{\partial}{\partial\bar{\psi}}\right) \ . \tag{B.13}$$

It is a bit weird to analyse the supersymmetric algebra here but here we note two things - firstly, the RHS of Eq. (B.13) shouldn't be interpreted as the Hamiltonian - we don't have time in zero-dimensions. The second point concerns with Eq. (B.12) - we see indeed $\mathcal{Q}^2$ is not zero in general but since $\psi\partial^2 W(x) = 0$ is the equation of motion the supersymmetric algebra indeed vanishes on-shell [33].

Localisation from coordinate transformations.

Let us first try and understand localisation from a coordinate transformation perspective. In particular, we would like to evaluate the path integral,

$$\mathcal{Z} = \int e^{-S} dx d^2\psi \ . \tag{B.14}$$

Firstly let us isolate the neighbourhoods $\mathcal{U}$ where the derivative superpotential ∂W vanishes. Taking the complement $\mathcal{U}^c$ of $\mathcal{U}$ in $\mathcal{F}$, we can change variables $(x,\psi,\bar{\psi}) \mapsto (y,\chi,\bar{\chi})$ where,

$$y = x - \frac{\psi\bar{\psi}}{\partial W} \ , \quad \chi = \psi\sqrt{\partial W} \ , \quad \bar{\chi} = \bar{\psi} \ . \tag{B.15}$$

The new measure is now,

$$dxd^2\psi = \sqrt{\partial W(y)}dyd^2\chi \ , \tag{B.16}$$

where $\mathcal{Q}(y) = 0 = \mathcal{Q}^\dagger(y)$ so y is invariant under supersymmetry [34]. We also see that the action transforms as,

$$S[y,0,0] = \frac{1}{2}\left(\partial W(y)\right)^2 = S[x,\psi,\bar{\psi}] \ . \tag{B.17}$$

[32] But with $\mathcal{Q}^\dagger(\psi) = -\delta W(x)$.

[33] This is the similar to the case in the free Wess-Zumino model hen we missed out degrees of freedom. Turns out we have simply missed out a bosonic auxiliary field - if we include such contribution, such as using the superfield formalism in $\mathbb{R}^{0|2}$, this will allow us to reproduce the full supersymmetry algebra with $\{\mathcal{Q},\mathcal{Q}\} = \{\mathcal{Q}^\dagger,\mathcal{Q}^\dagger\} = \{\mathcal{Q},\mathcal{Q}^\dagger\} = 0$.

[34] In fact y is the only independent combination of $(x,\psi,\bar{\psi})$ that is supersymmetrically invariant so any invariant function will be a function of y.

The contribution to the path integral is surprisingly,

$$\mathcal{Z}_{\mathcal{U}^c} = \frac{1}{2\pi}\int_{\mathcal{U}^c} e^{-S[y,0,0]}\sqrt{\partial W(y)}dyd^2\chi = 0 \ , \tag{B.18}$$

due to the property of the Berezin integral. This means that the non-vanishing contributions to $\mathcal{Z}$ only comes from the neighbourhood $\mathcal{U}$ - this is exactly where the coordinate transformation breaks down as $\partial W \to 0$ means the Jacobian of the coordinate transformation is no longer invertible. This leads us to the following key observation.

Proposition B.1 (Localisation principle.)**.** *Quantum field theories with supersymmetry generically have path integrals that localise to a vicinity of a fixed point set.*

How do we further evaluate this? Consider the case where W is a generic polynomial of degree d with $d-1$ isolated non-degenerate [35] critical points. Then around this critical point $x = x^*$, we can write,

$$W(x) = W(x^*) + \frac{\alpha_c}{2}(x - x^*)^2 + \dots \ , \tag{B.19}$$

with $\alpha_c = \partial^2 W(x^*)$. Then the action becomes,

$$S(x,\psi,\bar{\psi}) = \frac{\alpha_c^2}{2}(x - x^*)^2 - \alpha_c\bar{\psi}\psi \ , \tag{B.20}$$

and expanding the exponential in Grassmann variables in the integral will yield,

$$\begin{aligned}
\mathcal{Z} &= \sum_{x^*}\frac{1}{\sqrt{2\pi}}\int dxd^2\psi e^{-\frac{1}{2}\alpha_c^2(x-x^*)^2}\left(-1+\alpha_c\bar{\psi}\psi\right) \\
&= \sum_{x^*}\frac{\alpha_c}{\sqrt{2\pi}}\int e^{-\frac{1}{2}\alpha_c^2(x-x^*)^2} \\
&= \sum_{x^*}\frac{\alpha_c}{|\alpha_c|} \ ,
\end{aligned} \tag{B.21}$$

which eventually leads to,

$$\mathcal{Z} = \sum_{x^*:\partial W|_{x^*}=0}\frac{\partial^2 W(x^*)}{|\partial^2 W(x^*)|} \tag{B.22}$$

This is a surprising result. We note that if d is odd then $\mathcal{Z} = 0$, and if d is even then $\mathcal{Z} = \pm 1$ as we have $d-1$ critical points. $\mathcal{Z}$ just counts the number of times the superpotential crosses $W = 0$ [36]!

There is perhaps another way to illustrate this result. To do this we will need to discuss something know as deformation invariance.

[35] This means $\partial^2 W|_{x^*} \neq 0$.
[36] Or if you like, the number of kinks as a one-dimensional instanton.

B.2 Deformation Invariance

Deformation invariance can be summarised in one sentence: the path integral $\mathcal{Z}$ is sensitive only to the order of polynomial in W.

What do I mean by that? Let's suppose a quantum field theory has some symmetry G where it leaves the action and path integral measure invariant. Then the correlation functions of quantities that are variables of fields under the symmetry vanishes. To see this, let's suppose g is a field, and f is defined as the variation of ϕ under symmetry G,

$$f = \delta_G \phi \; , \tag{B.23}$$

Then the expectation value of f is,

$$\langle f \rangle = \int f e^{-S} = \int \delta_G g e^{-S} = \int \delta \left(g e^{-S} \right) = 0 \; . \tag{B.24}$$

For the present case, we can set,

$$g = \partial \rho(X) \bar{\psi} \; . \tag{B.25}$$

Now the variation of g under supersymmetry gives,

$$f = \epsilon \left(\partial \rho \partial W - \partial^2 W \psi \bar{\psi} \right) \; , \tag{B.26}$$

which leaves

$$\left\langle \partial \rho \partial W - \partial^2 W \psi \bar{\psi} \right\rangle = 0 \; . \tag{B.27}$$

Now since the action is,

$$S(X, \psi, \bar{\psi}) = \frac{1}{2} (\partial W)^2 - \psi \bar{\psi} \partial^2 W \; , \tag{B.7}$$

we can see that Eq. (B.27) gives the invariance of the correlation function,

$$\langle \delta_\rho S \rangle = 0 \; , \tag{B.28}$$

under the transformation of the superpotential $W \mapsto W + \rho$. This shows that the partition function is invariant under a change in the potential - which is true as long as ρ is small at infinity in field space when compared to h so the boundary terms in the argument will indeed vanish [37]. In particular, we can rescale $W \mapsto \lambda h$, with $\lambda \gg 1$. Then as long as,

$$\partial W e^{-\lambda^2 (\partial W)^2 / 2} \to 0 \tag{B.29}$$

when $|x| \to \infty$, boundary terms will not appear and the partition function $\mathcal{Z}$ will remain invariant. In particular, looking at the path integral $\mathcal{Z}$ now defined with this deformation parameter λ,

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int dx d^2 \psi e^{-S_\lambda} \; , \tag{B.30}$$

[37] ρ can be of the same order as h as long as the leading order term is smaller than that of h.

where the action is now,

$$S_\lambda(X,\psi,\bar{\psi}) = \frac{\lambda^2}{2}(\partial W)^2 - \lambda\psi\bar{\psi}\partial^2 W\ , \tag{B.31}$$

we see that, given the limit in Eq. (B.29) we will have,

$$\frac{d}{d\lambda}\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}}\int dx d^2\psi \mathcal{Q}^\dagger_\lambda\left(\psi\partial W e^{-S_\lambda}\right)\ , \tag{B.32}$$

which gives zero as no boundary terms will survive. The deformation invariance of path integral allows us to deduce that for $\lambda \to 0$, $e^{-\lambda^2(\partial W)^2/2}$ suppresses all contributions to the integral arbitrarily strongly apart from around the neighbourhood $\mathcal{U}$ of points x^* : $\partial W(x^*) = 0$. This is the other way to understand the localisation principle.

The deformation principle allow us to consider deformations of the superpotential $W(x)$. In particular, if $W(x)$ is a polynomial of order d, then we can deform $W(x)$ such that it has no critical points if d is odd and only one critical point if d is even - this is the same crossing phenomenon we have commented in the previous section, and we shall later see how this generalises to topological formulae in higher dimensions.

B.3 Explicit evalution

In fact, it is possible to evaluate directly the path integral $\mathcal{Z}$ in Eq. (B.14). We can write,

$$\begin{aligned}
\mathcal{Z} &= \frac{1}{\sqrt{2\pi}}\int dx d^2\psi e^{-S} \\
&= \frac{1}{\sqrt{2\pi}}\int dx \partial^2 W e^{-\frac{1}{2}(\partial W)^2} \\
&= \frac{D}{\sqrt{2\pi}}\int dy y e^{-\frac{1}{2}y^2} \\
&= D
\end{aligned} \tag{B.33}$$

where D is the degree of the map $x \mapsto y = \partial W(x)$. It enters the equation as the map is not one-to-one. The degree counts the number of preimages of a given point taking into account the relative orientation of each preimage with respect to its image so D is 0 and ± 1 respectively in the cases where d is odd and even, exactly as before.

One more side comment. A 'third' way of understanding localisation is to interpret the fermionic symmetry as some symmetry acting on the path integral G. In the most general case when G is freely acting, the integral over G just factors out (c.f. integration over an orbit in group theory like the Haar measure). The relevant integral here is $\int_G d\theta = 0$. However, our G, the group of fermionic symmetries parametrised by fermionic coordinate θ, has fixed locus $\mathcal{C}_0$ precisely in the open neighbourhood $\mathcal{U} \subset \mathcal{C}$ where $\mathcal{C} = \mathcal{F}$ is the space over which the integral is performed. Localisation exactly comes from these fixed points where the coordinate transformation is not well-defined as this is the fixed point of the fermionic symmetry G (generated by $\mathcal{Q}^\dagger$).

C Projective representations and covers

In the main text we have discussed how one should consider projective representations in QFTs. The central argument there is that instead of considering the projective representation, we can lift the group to the universal cover of the group and look at its 'normal' representations.

To properly discuss this requires a bit of topology set-up — in this section I will aim to provide a comprehensive account on what is needed to understand this theorem.

C.1 Covers and universal covers

We begin by defining what a covering map is [17].

Definition C.1. A subset $U \subset X$ is **evenly covered** by π if U is connected and open, and each component $\pi^{-1}(U)$ is an open set that is mapped homeomorphically onto U by π.

Definition C.2. A **covering map** is a continuous surjective map $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is path-connected and locally path-connected[38], and every point $p \in X$ has an evenly covered neighbourhood. We call $\tilde{X}$ the covering space of X and X the base of the covering.

Of course, everything so far is in the topology context. To specialise this to smooth manifolds (which is what we want), we will need to restrict the definition to a very specific type of covering map[39].

Definition C.3. Take E and M connected smooth manifolds with or without boundary. A map $\pi : E \to M$ is called a **smooth covering map** if π is smooth and surjective, and each point in M has a neighbourhood U such that each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U by π. We say U is evenly covered. We call M to be the **base manifold**, and E a **covering manifold of** M. If E is simply-connected, it is called the **universal covering manifold of** M.

Here *simply-connected* means every loop is path-isomorphic to a constant path [40]. We want to show that this universal covering exists and is in fact unique. I will here quote a few lemmas and theorems without detailed proof - the details can be found in the references [17, 24].

Theorem C.1. *Suppose M is a connected smooth-n-manifold, and $\pi : E \to M$ is a topological covering map. Then E is a topological n-manifold, and has a unique smooth structure such that π is a smooth covering map.*

[38] It might be surprising to see how path-connectedness does not generally imply locally path-connectedness. A counterexample is the topologist's sine curve, $y = \sin\left(\frac{1}{x}\right)$ for $x \in (0, \pi)$ together with closed arc connecting $(0, 0)$ and $(\pi, 0)$ where the space is path-connected but not locally path-connected at $(0, 0)$.

[39] By the way, if you are completely baffled by the definitions I just made, these are just mathematical details that you can skip (if you want, but I am weird so I will babble on). Alternatively you should pick up some topology books and start learning what topology is.

[40] Phrased in the language of the fundamental group at X, simply-connectedness simply means that the fundamental group of a manifold at every point $q \in M$ is the trivial group [17, 23].

Proof. See Proposition 4.40 of [17]. □

Corollary C.1. *If M is a connected smooth manifold, there exists a simply connected manifold $\tilde{M}$ - the* ***universal covering manifold of*** *M, and a smooth covering map $\pi : \tilde{M} \to M$. The universal covering manifold is unique such that for any other universal covering manifold $\tilde{M}'$ with projection map π', then there exists a diffeomorphism $\Phi : \tilde{M} \to \tilde{M}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. This is Corollary 4.43 of [17]. Since a proof is not given there I will give a sketch of the proof. The first step is show that any connected and locally simply connected space admits a unique universal cover. You will need some sort of path-connectedness arguments (see Theorem 12.8 of [24]) - to show the path classes are lifted to the upper space, then check several topological requirements (path-connectedness, topologies, covering maps). Now by Theorem C.1 you have the existence of a smooth covering manifold of M that is simply-connected. To show uniqueness, we need to find ϕ between any two universal covers that is a diffeomorphism - to show this find open sets such that you can find a surjective smooth submersion of $\pi\big|_{V^{-1}}$ must give you a smooth ϕ and ϕ^{-1} in both directions. □

Now it is straightforward to generalise this to Lie groups.

Theorem C.2. *Let G be a connected Lie group. There exists a simply connected Lie group $\tilde{G}$, called the universal covering group of G, that admits a smooth covering map $\pi : \tilde{G} \to G$ that is also a Lie group homomorphism.*

Proof. See Theorem 7.7 of [17]. Essentially the idea is you now need to also do group axiom checks on the universal covering group. □

Theorem C.3. *For any connected Lie group G, the universal covering group is unique in the following sense: if $\tilde{G}$ and $\tilde{G}'$ are connected Lie groups with corresponding smooth covering maps π and π', then there exists a Lie group homomorphism $\Phi : \tilde{G} \to \tilde{G}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. Again - this is similar to the proofs done above. See Theorem 7.9 of [17]. □

C.2 Projective representations

In the main text we have defined projective representations of G as the map $\rho : G \to PGL(V)$. Let me expand a bit on the definition stated above and make things a bit more precise. Recall in quantum mechanics we need states to be positive-definite to have a notion of probability in the Hilbert space — operators are therefore unitary or anti-unitary by Wigner's theorem. Then, we have the following definition.

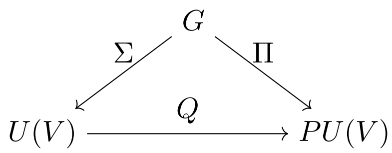
Definition C.4. Let $U(V)$ be the group of invertible linear transformations of a finite-dimensional Hilbert space V over $\mathbb{C}$ that preserve the inner product. A finite-dimensional **unitary representation** of a matrix Lie group G is a continuous homomorphism of $\Pi : G \to U(V)$ for some finite-dimensional Hilbert space V.

Definition C.5. Let V be a finite-dimensional Hilbert space over $\mathbb{C}$. The **projective unitary group** over V, denoted $PU(V)$ is then the quotient group

$$PU(V) = U(V)/e^{i\theta}I \tag{C.1}$$

where $e^{i\theta}I$ denotes the group of matrices in $U(1)I$, I being the identity matrix.

This establishes the codomain of the representation map. In particular, it can be shown that $PU(V)$ is isomorphic to a matrix Lie group[41]. Now let $Q : U(V) \to PU(V)$ be the quotient homomorphism and let $q : \mathfrak{u}(V) \to \mathfrak{pu}(V)$ be the associated Lie algebra isomorphism. We note that given an ordinary unitary representation $\Sigma : G \to U(V)$, we can always form a projective representation $\Pi : G \to PU(V)$ by setting $\Pi = Q \circ \Sigma$. This is equivalent to saying the following diagram commutes:



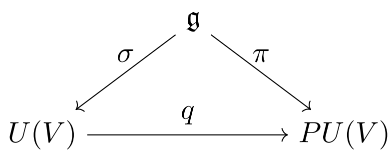
Note that not all projective representations arise in this fashion. I will state the following propositions without detailed proof.

Proposition C.1. *If V is a finite-dimensional Hilbert space over $\mathbb{C}$, then $PU(V)$ is isomorphic to a matrix Lie group. The associated Lie algebra homomorphism q defined above has the kernel $\{iaI\}$, so $PU(V)$ is isomorphic to $U(V)/\{iaI\}$.*

Proof. Consider the homomorphism $\Gamma : U(V) \to GL(gl(V))$, such that for given $U \in U(V)$, $\Gamma : U \mapsto C_U(X) = UXU^{-1}$. Then one can show that $\ker \Gamma = \{U(1)I\}$, so the image under this homomorphism is isomorphic to the quotient group $U(V)/\{e^{i\theta}I\}$, compact, and closed, i.e. a matrix Lie group isomorphic to $PU(V)$. To find the related Lie algebra homomorphism, we note that $c_X(Y) = [X, Y]$, with the kernel of c_X being the scalar multiples of I in $U(V)$ - the group $\{iaI\}$. The map c_X therefore must map onto $PU(V)$, giving the required isomorphism. □

Every finite-dimensional projective representation can be "de-projectivised" at the Lie-algebra level. To state this we have the following proposition.

Proposition C.2. *Let $\Pi : G \to PU(V)$ be a finite-dimensional projective unitary representation of a matrix Lie group G, and $\pi : \mathfrak{g} \to PU(V)$ be the associated Lie algebra homomorphism. Then there exists a Lie algebra homomorphism $\sigma : \mathfrak{g} \to U(V)$ such that $\pi(X) = q(\sigma(X)) \quad \forall X \in \mathfrak{g}$. So the following diagram commutes:*



[41]See Proposition 16.44 of [25].

This σ is unique upon fixing that $\operatorname{tr}\sigma(X) = 0 \quad \forall X \in \mathfrak{g}$.

Proof. This proposition boils down to the fact that you can always fix $\sigma(X)$ to have trace zero by choosing for $Y \in \mathfrak{u}(1)$, pick $\sigma(X) = Y + cI$ where c is an appropriate pure-imaginary constant. Such σ therefore always exist. (See Proposition 16.46 of [25] for more details.) □

Now we can say the most important theorem in this subsection:

Theorem C.4. *Suppose G is a matrix Lie group and $\tilde{G}$ is a universal cover of G with the covering map Φ. Then the following hold:*

1. *Let $\Pi : G \to PU(V)$ be a finite-dimensional projective unitary representation of G. Then there is an ordinary unitary representation $\Sigma : \tilde{G} \to U(V)$ of $\tilde{G}$ such that $\Pi \circ \Phi = Q \circ \Sigma$. Any such Σ is irreducible if and only if Π is irreducible. Σ is unique if we choose* $\det(\Sigma(A)) = 1,\ A \in \tilde{G}$.

2. *Let Σ be a finite-dimensional irreducible unitary representation of $\tilde{G}$. Then the kernel of the associated projective unitary representation $Q \circ \Sigma$ contains the kernel of the covering map Φ. Therefore $Q \circ \Sigma$ factors through G and gives rise to a projective unitary representation of G.*

Point 1 is equivalent to saying that the following box diagram commutes:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\Pi} & U(V) \\ \Phi \downarrow & & \downarrow Q \\ G & \xrightarrow{\Sigma} & PU(V) \end{array}$$

Proof. See Theorem 16.47 of [25]. The idea is actually really simple - we make use of Proposition C.2 to find an ordinary representation of $\mathfrak{g}$ at the base level, and then simply lift it up and apply Lie's Theorem at the cover level. The second half the theorem rests on the fact that $\ker\Phi$ is a discrete normal subgroup $\tilde{G}$ and is therefore central. We can then show that $\Sigma(A)$, where $A \in \ker\Phi$ under Schur's lemma gives $\Sigma(A) = cI$ as it intertwines V to itself. The A is in the kernel of the associated projective representation $Q \circ \Sigma$. □

Of course, we should be dealing with the infinite-dimensional case. Here of course the unitary representation needs to be defined slightly different[42] The main thing to note here is that we can no longer do the de-projectivisation by passing to the Lie algebra since there is no unique member we can choose - the notion of trace doesn't work for unbounded operators on the Hilbert space. Point 1 in Theorem **??** no longer works. However, if G is connected and "semi-simple", every projective unitary representation of G can be de-projectivised after passing to the universal cover. This is in fact the crucial reason why

[42]You will need some sort of a strong continuity homomorphism $\Phi : G \to U(\boldsymbol{H})$. You can read more about this in [25].

we need to study the universal covering manifolds! The spins intrinsically comes from this universal cover, and it is precisely since we are looking at the de-projectivised version of the representation that brought us there in the first place!

C.3 Central extensions

We are still not done. Recall the precisely statement we made in §2 is that there is a lifting of unitary representation of some projective Hilbert space, $\rho : G \to U(\mathbb{P}(\mathcal{H}))$ to the *central extension* of the universal covering group of the classical symmetry group. We haven't discussed anything about central extensions so far!

So let us properly address what we mean by the central extension of the universal covering group.

Definition C.6. An **extension of** G **by the group** A is given by an exact sequence of group homomorphisms,

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1\ . \tag{C.2}$$

The extension is **central** if A is abelian and its image $\mathrm{im}(\iota)$ is in the centre of E, i.e.

$$a \in A\ \ b \in E \implies \iota(a)b = b\iota(a) \tag{C.3}$$

Let us illustrate this by examples.

Example C.1. A **trivial extension** has the form,

$$1 \to A \xrightarrow{i} A \times G \xrightarrow{pr_2} G \to 1\ . \tag{C.4}$$

A non-trivial example is the following,

$$1 \to \mathbb{Z}/k\mathbb{Z} \to E = U(1) \xrightarrow{\pi} U(1) \to 1\ . \tag{C.5}$$

Example C.2 (Semi-direct products)**.** Recall that the **semidirect product** $G \ltimes H$ with a homomorphism $\tau : G \to \mathrm{Aut}(H)$ is given by,

$$(g, h)\ \cdot\ (g', h') = (gg', h\tau(g)(h'))\ . \tag{C.6}$$

This can be written as the extension,

$$1 \to H \xrightarrow{\iota} G \ltimes H \xrightarrow{\pi} G \to 1\ , \tag{C.7}$$

with $\pi(g, h) = h$ and $\iota(h) = (a, h)$ for fixed $a \in G$. For example, the semidirect group $GL(V) \ltimes V$ is isomorphic to the group of affine transformations.

Example C.3 (Lorentz group)**.** Obviously we can write,

$$1 \to \mathbb{Z}_2 \to SL(2, \mathbb{C}) \to SO(1, 3) \to 1\ , \tag{C.8}$$

where π is the two-to-one covering.

Example C.4. For a vector space V over field $\mathbb{F}$, we have,

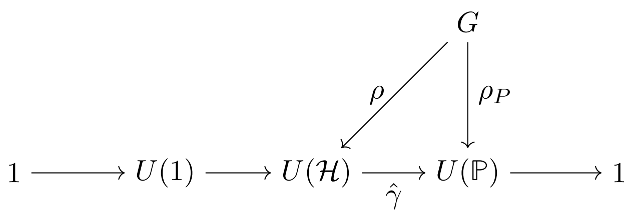
$$1 \to \mathbb{F}^* \xrightarrow{\iota} GL(V) \xrightarrow{\pi} PGL(V) \to 1 \; , \tag{C.9}$$

with $\iota : \lambda \mapsto \lambda \mathrm{id}_V$. Then $PGL(V)$ is the projective linear group which we have used in the definition of the projective representations.

Why do we need to set this up? Suppose $\mathcal{H}$ is a Hilbert space and $\mathbb{P}(\mathcal{H})$ is the projective space of 1d linear subspaces of $\mathcal{H}$, then we will have the central extension,

$$1 \to U(1) \xrightarrow{\iota} U(\mathcal{H}) \xrightarrow{\hat{\gamma}} U(\mathbb{P}) \to 1 \; . \tag{C.10}$$

From Wigner's theorem [43], we know that for every projective transformation $T \in \mathrm{Aut}(\mathbb{P})$ which are set of all projective transformations that preserve the transition probability, there exists a unitary or an anti-unitary operator $U \in U(\mathcal{H})$ where $T = \hat{\gamma}(U)$. So it is nice — we indeed have all the notions that we had before but now in the projective setting. But we could also ask: Given a projective representation T such that there is a continuous homomorphism $T : G \to U(\mathbb{P})$, does there exist a unitary representation $S : G \to U(\mathcal{H})$, such that the following diagram



commutes? Note that this is different from the above Wigner's theorem as this is not about the automorphisms of the projective Hilbert space $\mathbb{P}(\mathcal{H})$ but about representations! This turns out to be in general not achievable, and the key statement is the following.

Proposition C.3. *Given a representation $\rho_P : G \to U(\mathbb{P})$, there exists a lifting with respect to the central extension of the universal covering group of the classical symmetry group, $\tilde{\rho} : \hat{G} \to U(\mathcal{H})$.*

Proof. I will not discuss the proof and its glory details here, instead you can find the details in Theorem 3.10 in [8] which gives a natural lift of the representation by a $U(1)$ extension, and then use Bargmann's Theorem which states that every connected and simply-connected G admits such a lift to finish the proof. □

The short form is the following. Since the universal cover of G is just the central extension by the fundamental group $\pi_1(G)$, we can always apply Proposition C.3 to work with the universal cover and apply Bargmann's Theorem. This is merely a restatement of the result in Theorem C.4 — the results don't change, but merely highlights the need to work with universal covering groups in physics so we have a chance of working with $U(\mathcal{H})$. Phew!

[43] See the Appendix of [8] for the proof.

D Clifford algebras and spinors

This section deals with Clifford algebras and spinor representations properly. It is quite mathematically heavy, so please look away if you find maths horrifying.

D.1 Real, imaginary and quaternionic representations

Before we begin let us address some notions about representations. Fix G a group.

Definition D.1. Let V be a vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. A **Hermitian form** on V is an $\mathbb{F}$-bilinear form $V \times V \to \mathbb{F}$ such that for all $v_1, v_2 \in V$ and $\lambda \in \mathbb{F}$,

$$\langle v_1, \lambda v_2 \rangle = \lambda \langle v_1, v_2 \rangle \ , \quad \langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle} \ . \tag{D.1}$$

Suppose G is a finite group , a compact Lie group or a semisimple Lie group. Then V has a G-invariant non-degenerate hermitian form. If $|G| = \infty$ the proof is a bit more complicated. It turns out that Clifford algebras is almost the group algebra of a finite group, so very similar arguments apply there.

Definition D.2. Let V be a complex vector space. A linear map $\varphi : V \to V$ is a **real (quaternionic)** structure if φ obeys the following conditions:

1. φ is conjugate linear, $\varphi(\lambda v) = \bar{\lambda}\varphi(v)$ for $\lambda \in \mathbb{C}$ and $v \in V$,
2. $\varphi^2 = 1$ (-1 in the quaternionic case).

We then have the following lemma.

Lemma D.1. *Let V be a complex vector space and $c : V \to V$ be a real structure. Then $V = V_+ \oplus V_-$ where $V_\pm$ are isomorphic real vectors spaces, or $V \cong \mathbb{C} \otimes V_+$.*

Proof. $c^2 = 1$ implies the eigenvalues of c are ± 1; and since c is conjugate linear $V_\pm$ are real subspaces. The isomorphism map is given by $i : V_+ \to V_-$. □

The real structure is simply a notion of complex conjugation. A quaternionic structure however allows us to define a left action $\mathbb{H}$ on V. In particular, suppose $J : V \to V$ is a quaternionic structure, then $J^2 = -1$ and $-iJ = Ji$, so if $q = a + bj$ is a quaternion then $qv = av + bJ(v)$ for $v \in V$.

Definition D.3. Let V be a complex representation of G. We say that V is of **real (quaternionic)** type if V processes a G-invariant real (quaternionic) structure.

Theorem D.1. *A complex representation V of G is of real (quaternionic) type iff V admits a non-degenerate symmetric (anti-symmetric) G-invariant complex bilinear form $B : V \times V \to \mathbb{C}$.*

Proof. The proof is a bit complicated and requires a bit of set-up. Let $B : V \times V \to \mathbb{C}$ a non-degenerate complex bilinear G-invariant map that satisfy $B(v_1, v_2) = \epsilon B(v_2, v_1)$ with $\epsilon = \pm 1$. With a G-invariant hermitian form $\langle \cdot, \cdot \rangle$ on V, we can define,

$$B(v_1, v_2) = \langle \varphi(v_1), v_2 \rangle \ , \tag{D.2}$$

for some $\varphi : V \to V$. This is conjugate linear, G-invariant and an isomorphism. It can be shown that [26] φ can be rescaled to give the required structure map. Conversely, given the structure map $\mathcal{J} : V \to V$ satisfying $\mathcal{J}^2 = \epsilon \mathbb{1}$, we can take the G-invariant symmetric non-degenerate $\mathbb{R}$-bilinear form and extend by complex linearity to a non-degenerate G-invariant symmetric $\mathbb{C}$-bilinear form on V. The rest of the details are in [26]. □

Let us illustrate this using some basis. Firstly, we can write some Hermitian form using a basis to a Hermitian matrix A,

$$\langle u, v \rangle = \bar{u}^T A v \ . \tag{D.3}$$

Let g be the matrix representing the group action of $g \in G$. Then, the G-invariance of the Hermitian form means,

$$\bar{g}^T \cdot A \cdot g = A \ . \tag{D.4}$$

Similarly, a bilinear form can be represented by a matrix B,

$$B(u, v) = u^T B v \ , \tag{D.5}$$

and G-invariance demands,

$$g^T \cdot B \cdot g = B \ . \tag{D.6}$$

Let us now illustrate how real and quaternionic structures can be represented. For $V \cong \mathbb{C}^n$ a complex vector space, B a non-degenerate $\mathbb{C}$-bilinear form can be represented by $B^T = \epsilon B$. Now the structure map $\mathcal{J} : V \to V$ is conjugate linear so can only be represented by some real basis. So we look at the underlying real vector space with a complex structure $I : V_{\mathbb{R}} \to V_{\mathbb{R}}$, given by

$$I_{\mathbb{R}} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \tag{D.7}$$

and the matrices A and B are represented by,

$$A_{\mathbb{R}} = \begin{pmatrix} A & -iA \\ iA & A \end{pmatrix} \ , \quad B_{\mathbb{R}} = \begin{pmatrix} B & iB \\ iB & -B \end{pmatrix} \ . \tag{D.8}$$

Now since we want $B(u, v) = \langle \mathcal{J}(u), v \rangle$. We can take the conjugate to get $\epsilon \overline{B(v, u)} = \langle v, \mathcal{J}(u) \rangle$, and in a basis we will find that,

$$A_{\mathbb{R}} \cdot J_{\mathbb{R}} = \epsilon \bar{B}_{\mathbb{R}} \ . \tag{D.9}$$

Since $A_{\mathbb{R}}$ and $B_{\mathbb{R}}$ are invertible, this constrains $J_{\mathbb{R}}$ and we will get at the end,

$$J_{\mathbb{R}} = \epsilon A_{\mathbb{R}}^{-1} \bar{B}_{\mathbb{R}} \ . \tag{D.10}$$

D.2 Clifford Algebras and Gamma Matrices

Let us now construct Clifford algebras. The following discussion mainly follows [27].

D.2.1 Clifford algberas, abstractly

We begin with a definition. In this section we fix V to be a vector space and $B : V \times V \to \mathbb{K}$ to be a symmetric bilinear form on V (here $\mathbb{K}$ is a field). To construct a corresponding quadratic form Q, we can define

$$Q(x) = B(x, x) \ , \tag{D.11}$$

such that (V, Q) is a **quadratic vector space over** $\mathbb{K}$. One can conversely reconstruct the symmetric bilinear form B from Q by polarisation, i.e.

$$B(x, y) = \frac{1}{2} \left(Q(x + y) - Q(x) - Q(y) \right) \ , \tag{D.12}$$

so we hereby denote Q and B interchangeably, writing $Q(x) = Q(x, x)$. Let us also define what an associative algebra is.

Definition D.4. An **associative algebra over** R is a ring A together with a ring homomorphism from R into the centre of A, i.e. for $r \in R$ and $a, b \in A$ then,

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y) \ . \tag{D.13}$$

Now we have the following definition. I have here chosen the sign conventions to conform with [10, 11, 26].

Definition D.5. Let A be an associative $\mathbb{K}$-algebra and (V, Q) be a quadratic vector space. A $\mathbb{K}$-linear map $\phi : V \to A$ is **Clifford** if $\forall x \in V$,

$$\phi(x)^2 = -Q(x) 1_A \ , \tag{D.14}$$

where 1_A is the unit of A.

Definition D.6. The **Clifford algebra** $C = C(Q) = \mathrm{Cliff}(V, Q)$ is an associative algebra with unit 1 and is generated by V such that for all $v \in V$ we have,

$$v \cdot v = -Q(v, v) \cdot 1 \ . \tag{D.15}$$

Equivalently (if the characteristic of k is not 2), we have $\forall v, w \in V$,

$$v \cdot w + w \cdot v = -2Q(v, w) \ . \tag{D.16}$$

A note about construction. The Clifford algebra can be constructed quickly by taking the tensor algebra,

$$T^\bullet(V) = \bigoplus_{n \geq 0} V^{\otimes n} \ , \tag{D.17}$$

and setting

$$C(Q) = \frac{T^\bullet(V)}{I(Q)} \ . \tag{D.18}$$

Here $I(Q)$ is the two-sided ideal generated by all elements of the form $v \otimes v - Q(v, v) \cdot 1$. Clearly $C(Q)$ satisfies the universal property. From this we can see that the dimension of C is 2^m where $m = \dim(V)$ and that the canonical mapping $V \to C$ is an embedding, with the basis of $C(Q)$ being the products $e_I = e_{i_1} \cdot e_{i_2} \cdot ... \cdot e_{i_k}$ where e_i are the basis of V. To see this, in particular, we can check the following.

Proposition D.1. *There is a natural embedding* $V \hookrightarrow C(Q)$ *which is the image of* $V = V^{\otimes 1}$ *under the canonical projection*

$$\pi_q : T^\bullet(V) \to C(Q) \ , \tag{D.19}$$

and this is an injection.

Proof. Say that an element $\varphi \in T^\bullet(V)$ is of pure degree s if $\varphi \in V^{\otimes s}$. We want to show that any element $\varphi \in T^\bullet(V) \cap V$ is zero. Suppose this is not true. Then we can write $\varphi = \sum_i a_i \otimes (v_i \otimes v_i + Q(v_i)) \otimes b_i$ where we assume that a_i and b_i is of pure degree. Now since $\varphi \in V$ we must have that the expression is equal to zero, with the sum taken over those indices with $\deg a_i + \deg b_i$ maximal. Contracting with Q means $\sum_i a_i Q(v_i) \cdot b_i = 0$. Proceed with induction to show $\varphi = 0$. □

The Clifford Algebra has a universal property as follows. This also gives a categorical definition of Clifford Algebras.

Proposition D.2. *The Clifford algebra can be defined to be the universal algebra with the following property: If* A *is any associative algebra with unit and a linear mapping* $j : V \to A$ *is given such that*

$$j(v) \cdot j(v) = -Q(v, v) \cdot 1, \quad \forall v \in V \ , \tag{D.20}$$

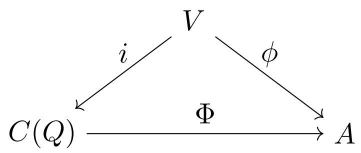
or equivalently $\forall v, w \in V$*, (given that* k *has a characteristic not equal to 2,)*

$$j(v) \cdot j(w) + j(w) \cdot j(v) = -2Q(v, w) \cdot 1 \ . \tag{D.21}$$

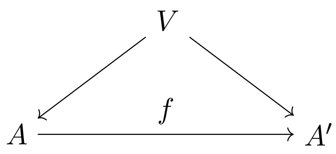
then there should be a unique homomorphism of algebras from $C(Q)$ *to* A *extending* j*, i.e.* j *extends uniquely to a* $\mathbb{K}$*-algebra homomorphism* $\tilde{j} : C(Q) \to A$*, and* $C(Q)$ *is the unique associative* $\mathbb{K}$*-algebra with this property.*

Proof. Any linear map $j : V \to A$ extends to a unique algebra homomorphism $\bar{j} : T^\bullet(V) \to A$. Now Eq. (D.20) implies that $\bar{j} = 0$ on $I(Q)$ so $\bar{j}$ descends to $C(Q)$. Suppose now B is an associative $\mathbb{K}$-algebra with unit and that $\iota : V \to B$ is an embedding with the property that any linear map $j : V \to A$ with the property in Eq. (D.20) extends uniquely to an algebra homomorphism $\tilde{j} : A \to B$. Then the isomorphism from $V \subset Q$ to $\iota(V) \subset B$ clearly induces an algebra isomorphism $C(Q) \xrightarrow{\cong} B$. □

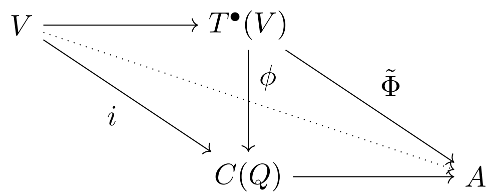
The proposition above effectively states the following. Given an associative algebra with unit A, together wiht a Clifford map $i : V \to C(Q)$ such that for every Clifford map $\phi : V \to A$ there is a unique algebra morphism $\Phi : C(Q) \to A$ that makes the following triangle commute.



Categorically, the Clifford Algebra is an initial object in the category **Cliff**(V, Q), which has Clifford maps $\phi : V \to \cdot$ from a fixed vector space equipped with a quadratic form Q as objects. The morphism from $V \to A$ to $V \to A'$ is given by a commuting triangle



with $f : A \to A'$ as a homomorphism of associative algebras. This initial object is unique up to a unique isomorphism. In other words, the Clifford algebra $C(Q)$ is universal for Clifford maps to associative algebras. The construction via tensor algebra as before implies the following statement. If $\phi : V \to A$ is a Clifford map and $\tilde{\Phi} : T^{\bullet}(V) \to A$ is the unique extension of ϕ to the tensor algebra, then $\tilde{\Phi}$ indeed annihilates the ideal $I(Q)$ and therefore factors through a unique map $\Phi : T^{\bullet}(V)/I(Q) \to A$ from the quotient. Therefore, we have a commutative diagram:



Here i is really injective as the ideal only comes into play for $V^{\geq \otimes 2}$.

D.2.2 Constructing Clifford algebras

The way we have been discussing about Clifford algebras is not very suitable for computations. Instead, we will discuss the way that Clifford introduced the algebras. This is the way Clifford algebras are still taught in physics courses, following Dirac.

Traditionally, the discussion of Clifford algebras started with Dirac matrices.

Definition D.7. Suppose $\{e_i\}$ is a $\mathbb{K}$-basis for V, where $i = 1, ..., \dim V$. The vector space V is equipped with the symmetric bilinear form where $B(e_i, e_j) = B_{ij} = B_{ji}$. The **Clifford generators** Γ_i is the image of e_i under the map $i : V \to C(Q)$, which satisfy the relations

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2B_{ij}\mathbb{1} \tag{D.22}$$

where $\mathbb{1}$ is the unit in the Clifford algebra $C(Q)$.

Following this, we can define Clifford algebras by using generators in the following manner.

Definition D.8 (Clifford algebras — generators)**.** An associative algebra over field $\mathbb{K}$ with unity 1 is the **Clifford algebra** $C(Q)$ of a non-degenerate quadratic form Q on V if it contains V and $\mathbb{K} = \mathbb{K} \cdot 1$ as distinct subspaces such that the following three conditions hold:

(i) $v^2 = Q(v)$ for any $v \in V$.

(ii) V generates $C(Q)$ as an algebra over $\mathbb{K}$.

(iii) $C(Q)$ is not generated by any proper subspace of V.

We can immediately how the Dirac matrices furnishes a representation of the Clifford algebra. We then define

$$\Gamma_{ij} = \frac{1}{2} \left(\Gamma_i \Gamma_j - \Gamma_j \Gamma_i\right) \ , \tag{D.23}$$

as the product of two generators. More generally, we have

$$\Gamma_{i_1 \dots i_p} = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma \Gamma_{i_{\sigma(1)}} \dots \Gamma_{i_{\sigma(p)}} \ , \tag{D.24}$$

where $(-1)^\sigma$ indicates the sign of the permutation in S_p. We then see that since $C(Q)$ is generated by V and the identity it is the linear span of $\mathbb{1}$, Γ_i, Γ_{ij},... in total there are $1 + n + C_2^n + \dots C_n^n = 2^n$ monomials. So $\dim C(Q) = 2^{\dim V}$. This is the same dimension as the exterior algebra $\mathrm{Ext}^\bullet V$ so we can establish a vector space isomorphism between the two.

In particular, if we use an orthonormal basis to generate $C(Q)$, then the first condition in the above Definition D.8 then becomes

$$\Gamma_i^2 = 1, \quad 1 \leq i \leq p, \tag{D.25}$$

$$\Gamma_i^2 = -1, \quad p < i \leq p, \tag{D.26}$$

$$\Gamma_i \Gamma_j = -\Gamma_j \Gamma_i, \quad i < j. \tag{D.27}$$

whilst condition (iii) becomes

$$\Gamma_1 \dots \Gamma_n \neq \pm 1 \ . \tag{D.28}$$

This is important in constructing a representation of Clifford algebras in general dimensions. Typically, this is needed in the discussion of supersymmetry and supergravity (and spinors) in various dimensions. The construction typically involves a set of matrices called **Dirac matrices** or **Gamma matrices**, defined as matrix representations of the Clifford algebra in various dimensions. You should have seen **Pauli matrices** in your elementary quantum mechanics courses:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{D.29}$$

These matrices can generate a basis for Clifford algebras of arbitrary dimensions. Here we follow the discussion in [9, 28, 29]. We construct the Euclidean γ-matrices from Gamma matrices which are the basic building block of the matrix representations of the Clifford algebras. We define $(2k+1)$-matrices by the tensor products of k Pauli matrices to get a $2^k \times 2^k$ matrix representation as follows:

$$\begin{aligned}
\Gamma_1^{(k)} &= \sigma_1 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-1} , \quad \Gamma_2^{(k)} = \sigma_2 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-1} , \\
\Gamma_3^{(k)} &= \sigma_3 \otimes \sigma_1 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-2} , \quad \Gamma_4^{(k)} = \sigma_3 \otimes \sigma_2 \otimes \underbrace{\sigma_0 \otimes ... \otimes \sigma_0}_{k-2} , \\
&\vdots \\
\Gamma_{2k-1}^{(k)} &= \underbrace{\sigma_3 \otimes ... \otimes \sigma_3}_{k-1} \otimes \sigma_1 , \quad \Gamma_{2k}^{(k)} = \underbrace{\sigma_3 \otimes ... \otimes \sigma_3}_{k-1} \otimes \sigma_2 , \\
\Gamma_{2k+1}^{(k)} &= \underbrace{\sigma_3 \otimes ... \otimes \sigma_3}_{k}
\end{aligned} \tag{D.30}$$

The matrices listed above can be generated using the recurring relations:

$$\Gamma_M^{(k+1)} = \Sigma_M^{(k)} \otimes \sigma_0 , \quad M = 1, ..., 2k \tag{D.31}$$

$$\Gamma_{2k+i}^{(k+1)} = \Sigma_{2k+1}^{(k)} \otimes \sigma_i , \quad i = 1, 2, 3 \tag{D.32}$$

which gives

$$\left\{ \Gamma_M^{(k)}, \Gamma_N^{(k)} \right\} = -2\delta_{MN} \tag{D.33}$$

So then we have the following definition.

Definition D.9 (Gamma matrices)**.** The **Gamma** or **Dirac matrices** are matrix representations of the Clifford algebras, i.e. the map: $\Gamma : C(Q) \to \mathrm{GL}(\mathbb{C}^{2^k})$ where we map the generators $e_M \mapsto \Gamma_M$. The representation is faithful when $d = 2k$ and non-faithful when $d = 2k+1$ where $\Gamma(\epsilon) = \Gamma_1^{(k)} ... \Gamma_{2k+1}^{(k)} = i^k$.

We will find Gamma matrices extremely helpful later when we construct spinors in spaces of Euclidean and Lorentzian signatures. In particular, for Lorentzian gamma matrices we will need to do some modifications. To obtain these, we pick some single matrix from the Euclidean construction, multiply by i and label it as Γ_0. This matrix is anti-Hermitian and satisfies,

$$\Gamma_0^2 = -\mathbb{1} . \tag{D.34}$$

We relabel the remaining set of gamma matrices to obtain the other gamma matrices. Then the Hermitian properties of the Lorentzian gamma matrices are then given by,

$$\Gamma_M^\dagger = \Gamma_0 \Gamma_M \Gamma_0 . \tag{D.35}$$

There is a caveat — these gamma matrices are identified up to a conjugacy class,

$$\Gamma_M \sim S \Gamma_M S^{-1} , \tag{D.36}$$

in particular, we can choose a unique irrep which is Hermitian for even dimensions and for odd dimensions there are two mathematically inequivalent irreducible representations which differ only by the sign of the final gamma matrix Γ_{2N+1}.

D.2.3 $\mathbb{Z}_2$-grading and exterior algebras

Let us return to the tensor construction of the Clifford algebras. Since the ideal $I(Q)$ is not homogeneous, $C(Q)$ does not inherit a $\mathbb{Z}$-grading from $T^\bullet(V)$. However, notice that the ideal $I(Q)$ is generated by elements of an even degree. This means the Clifford algebra does inherit a $\mathbb{Z}_2$ grading. To study this grading recall the following definitions from elementary algebra.

Definition D.10. A **graded ring** is a ring that is decomposed into a direct sum of additive groups

$$R = \bigoplus_{n=0}^{\infty} R_n = R_0 \oplus R_1 \oplus R_2 \oplus \dots \ , \tag{D.37}$$

such that

$$R_m R_n \subseteq R_{m+n} \ , \tag{D.38}$$

for all non-negative integers m and n.

Definition D.11. An associative algebra A over a ring R is **graded** if it is graded as a ring.

So we can now go back to Clifford Algebras. Consider the automorphism $\alpha : C(Q) \to C(Q)$ which sends $\alpha(v) = -v$ on V. Since $\alpha^2 = \mathrm{id}$, the ideal $I(Q)$ is generated by elements of an even degree, and hence Clifford algebra inherits a $\mathbb{Z}_2$ grading:

$$C(Q) = C^0(Q) \oplus C^1(Q) \ , \tag{D.39}$$

where $C^i(Q) = \left\{\varphi \in C(Q) \,\middle|\, \alpha(\varphi) = (-1)^i \varphi\right\}$ are the eigenspaces of α. Since α is a homomorphism, we have

$$C^i(Q) \cdot C^j(Q) \subset C^{i+j}(Q) \ , \tag{D.40}$$

with the indices taken modulo 2. This $\mathbb{Z}_2$-grading plays an important role in the analysis and application of Clifford algebras. In particular, $C^0(Q)$ is often called $C^{\mathrm{even}}(Q)$ and is a subalgebra of dimension 2^{m-1}, where as $C^1(Q)$ is often called $C^{\mathrm{odd}}(Q)$.

The $\mathbb{Z}_2$-gradedness of the Clifford algebra is very different from the graded nature of the tensor algebra which inherently has a $\mathbb{Z}$-graded structure. To see this, define $\tilde{\mathcal{F}}$ as

$$\tilde{\mathcal{F}}^r = \sum_{s \leq r} V^{\otimes s} \ . \tag{D.41}$$

This has the property

$$\tilde{\mathcal{F}}^r \otimes \tilde{\mathcal{F}}^s \subset \tilde{\mathcal{F}}^{r+s} \ . \tag{D.42}$$

The tensor algebra therefore has a natural filtration

$$\tilde{\mathcal{F}}^0 \subset \tilde{\mathcal{F}}^1 \subset \dots \subset T^\bullet(V) \ , \tag{D.43}$$

which makes the tensor algebra into a **filtered algebra**. Every filtered algebra has an associated graded algebra. For the tensor algebra with the canonical filtration described above, the associated graded algebra is described by

$$\tilde{\mathcal{G}}^p = \tilde{\mathcal{F}}^p / \tilde{\mathcal{F}}^{p-1} \,. \tag{D.44}$$

Then $\tilde{\mathcal{G}}^\bullet$ is a graded algebra where the product map is defined by

$$\tilde{\mathcal{G}}^p \times \tilde{\mathcal{G}}^q \to \tilde{\mathcal{G}}^{p+q} \,. \tag{D.45}$$

The canonical filtration of the tensor algebra $T^\bullet(V)$ defines a natural filtration on the Clifford algebra $C(Q)$. Suppose $\pi_q : T^\bullet(V) \to T^\bullet(V)/I(Q)$ where $I(Q)$ is the ideal that generates the Clifford algebras. Then $\mathcal{F}^i = \pi_q(\tilde{\mathcal{F}}^i)$ naturally has a natural filtration,

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \ldots \,, \tag{D.46}$$

and naturally the associated graded algebra $\mathcal{G}^r = \mathcal{F}^r / \mathcal{F}^{r-1}$ naturally inherits the filtration. We now have the following proposition.

Proposition D.3. *For any quadratic form Q, the associated graded algebra of $C(Q)$ is naturally isomorphic to the exterior algebra* $\mathrm{Ext}^\bullet V$.

Proof. The map $\bigotimes^r V \xrightarrow{\pi_r} \mathcal{F}^r \to \mathcal{G}^r = \mathcal{F}^r/\mathcal{F}^{r-1}$ given by $v_{i_1} \otimes \ldots \otimes v_{i_r} \mapsto [v_{i_1} \ldots v_{i_r}]$ descends to a map $\mathrm{Ext}^r V \to \mathcal{F}^r$ by the property in Eq. (D.21). (Note that when the characteristic of $\mathbb{K}$ is 2 then we will have to use the other condition.) This map is surjective and gives a homomorphism of graded algebras $\mathrm{Ext}^\bullet V \to \mathcal{G}^\bullet$. It remains to check the map is injective. The kernel of $\bigotimes^r V \to \mathcal{G}^r$ consists of the r-homogeneous pieces of elements $\varphi \in I_q(V)$ of degree less than r. Any such φ can be written as a finite sum $\varphi = \sum a_i \otimes (v_i \otimes v_i + q(v_i)) \otimes b_i$ where $v_i \in V$ and where we may assume that the a_i and b_i are of pure degree with $\deg a_i + \deg b_i \leq r-2$. The r-homogeneous part of φ is then of the form $\varphi_r = \sum a_i \otimes v_i \otimes v_i \otimes v_i$ where $\deg a_i + \deg b_i = r-2$ for each i. The image of φ in the exterior algebra is however zero as $v_i \wedge v_i = 0$. So the map $\mathrm{Ext}^r V \to \mathcal{G}^r$ is injective. ☐

Note that the proposition above gives a canonical vector space isomorphism that is compatible with the filtrations as follows,

$$\mathrm{Ext}^\bullet V \to C(Q) \,. \tag{D.47}$$

The map in Eq. (D.47) is of course not an isomorphism of algebras unless $q = 0$. However the map is indeed canonical so we can discuss embeddings of the form $\mathrm{Ext}^r V \subset C(Q)$ for all $r \geq 0$. To see that the isomorphism is only true when $q = 0$, consider the $\mathbb{Z}_2$-grading on the tensor algebra defined with $T^\bullet(V) = T^\bullet(V)_0 + T^\bullet(V)_1$ where

$$T^\bullet(V)_0 = \bigoplus_{k\geq 0} V^{\otimes 2k} \,, \quad T^\bullet(V)_1 = \bigoplus_{k\geq 0} V^{\otimes 2k+1} \,. \tag{D.48}$$

where the $\mathbb{Z}_2$-grading is the reduction mod-2 of the $\mathbb{Z}$-grading of the tensor algebra as discussed above. This reduction makes the ideal I_q homogeneous, and hence the projection

$T^\bullet(V) \to C(Q)$ restricts to projections $TV_i \to C_i$ for $i = 0, 1$. Note however that for $i = 1$ this is only a projection of vector spaces, since neither TV_1 nor C_1 are algebras.

Now the canonical filtration of the tensor algebra $T^\bullet(V)$ defines a filtration on $C(Q)$ as follows. By filtering $T^\bullet(V)_0$ and $T^\bullet(V)_1$ separately, i.e.

$$\mathcal{F}^{2k}T(V)_0 = \bigoplus_{l \leq k} V^{\otimes 2l} \ , \quad , \mathcal{F}^{2k+1}T(V)_1 = \bigoplus_{l \leq k} V^{\otimes 2l+1} \tag{D.49}$$

such that

$$0 \subset \mathcal{F}^0 T(V)_0 \subset \mathcal{F}^2 T(V)_0 \subset ... \ , \tag{D.50}$$

$$0 \subset \mathcal{F}^1 T(V)_1 \subset \mathcal{F}^3 T(V)_1 \subset ... \ . \tag{D.51}$$

Now under the projections $TV_0 \to C_0$ and $TV_1 \to C_1$, we can similarly identify the filtrations of the Clifford algebra as

$$0 \subset \mathcal{F}^0 C_0 \subset \mathcal{F}^2 C_0 \subset ... \ , \tag{D.52}$$

$$0 \subset \mathcal{F}^1 C_1 \subset \mathcal{F}^3 C_1 \subset ... \ . \tag{D.53}$$

We will henceforth use the shorthand $\mathcal{F}^p C$ as $\mathcal{F}^p C_0$ and $\mathcal{F}^p C_1$ if p is even and odd respectively. Now we note that $\mathcal{F}^p C / \mathcal{F}^{p-2} C \cong \operatorname{Ext}^p V$ as the corrections in replacing xy by $-yx$ where $x, y \in V$ involve terms of degree 2 less. The corrections are 0 when $q = 0$, so we can identify $C(Q) \cong \operatorname{Ext}^\bullet V$, exactly as mentioned above.

It is possible to understand the relation between the Clifford and exterior algebras in a different way which does not involve filtrations. The bilinear form B defines a map $\flat : V \to V^*$ where $x \mapsto B(x, \cdot)$. The map $flat$ is an isomorphism if and only if B is non-degenerate. The inverse is typically defined as $\sharp$ so together with the map $\flat$ they are referred to as the musical isomorphisms induced from the inner product B. We can then define a linear map $\phi : V \to \operatorname{End}(\operatorname{Ext}^\bullet V)$ by

$$\phi(x)\alpha = x \wedge \alpha \iota_x \alpha \tag{D.54}$$

where ι_x is the unique odd derivation defined by $\iota_x 1 = 0$ and $\iota_x y = B(x, y)$ for $y \in V$. So on a monomial we have,

$$\iota_x \left(y_1 \wedge ... \wedge y_p \right) = \sum_{i=1}^{p} (-1)^{i-1} B(x, y_i) y_1 \wedge ... \wedge \hat{y_i} \wedge ... \wedge y_p \ , \tag{D.55}$$

where the hat denotes omission. Then we can extend this linearly to all of $\operatorname{Ext}^\bullet V$ as in the following lemma.

Lemma D.2. *The map* $\phi : V \to \operatorname{End} V$ *in Eq.* (D.54) *is Clifford.*

Proof. For every $x \in V$ and $\alpha \in \operatorname{End} \operatorname{Ext}^\bullet V$, we have

$$\begin{aligned} \phi(x)^2 \alpha &= \phi(x) \left(x \wedge \alpha - \iota_x \alpha \right) \\ &= x \wedge x \wedge \alpha - x \wedge \iota_x \alpha - Q(x)\alpha + x \wedge \iota_x \alpha + \iota_x \iota_x \alpha \\ &= -Q(x)\alpha \ , \end{aligned} \tag{D.56}$$

where $x \wedge x = 0 = \iota_x \iota_x$ and $\iota_x (x \wedge \alpha) = Q(x)\alpha - x \wedge \iota_x \alpha$. □

By the universality of Clifford algebras we can then extend this to the algebra homomorphism uniquely,

$$\Phi : C(Q) \to \operatorname{End}\operatorname{Ext}^\bullet V \ . \tag{D.57}$$

So composing this with the evaluation at $1 \in \operatorname{Ext}^\bullet V$ gives a linear map $\Phi_1 : C(Q) \to \operatorname{Ext}^\bullet V$. This map obeys $\Phi_1(1) = 1$ and if $x \in V$ then $\Phi_1(i(x)) = x$ where $i : V \to C(Q)$. Since i is injective from the construction of $C(Q)$, $\Phi_1 \circ i$ is also injective. By further computations, we then get

$$\Phi_1(i(x)i(y)) = x \wedge y - B(x, y) \ , \tag{D.58}$$

and

$$\Phi_1(i(x)i(y)i(z)) = x \wedge y \wedge z - B(x, y)z + B(x, z)y - B(y, z)x \ , \tag{D.59}$$

so Φ_1 surjects onto $\operatorname{Ext}^\bullet V$. This is a vector space isomorphism with the inverse map defined by

$$y_1 \wedge ... \wedge y_p \mapsto \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma y_{i_{\sigma(1)}} \wedge ... \wedge y_{i_{\sigma(p)}} \ , \tag{D.60}$$

which gives an explicit quantisation of the exterior algebra.

D.2.4 Clifford algebras as representations of $\mathfrak{so}_n\mathbb{C}$

It is clear from the construction that Clifford algebras are associative algebras. As a results it determines a Lie algebra with the bracket defined by the associative multiplication. How are Clifford algebras related to the representations of $\mathfrak{so}_m\mathbb{C}$? To see this we will first embed the Lie algebra $\mathfrak{so}(Q)$ inside the Lie algebra of the even part of the Clifford algebra and from there identify $C(Q)$ with one or two copies of matrix algebras.

Let us see how this works in practice. We first need to make an explicit isomorphism of $\operatorname{Ext}^2 V$ with $\mathfrak{so}(Q)$, which is defined as

$$\mathfrak{so}(Q) = \{X \in \operatorname{End}(V) \,|\, Q(Xv, w) = -Q(v, Xw) \ \forall v, w \in V\} \ . \tag{D.61}$$

Define the map

$$\varphi_{a\wedge b}(v) = 2\left(Q(b, v)a - Q(a, v)b\right) \ , \tag{D.62}$$

which gives the isomorphism $\phi : \operatorname{Ext}^2 V \to \mathfrak{so}(Q) \subset \operatorname{End}(V)$ with $a \wedge b \mapsto \varphi_{a\wedge b}$. One can check that the bracket on $\operatorname{Ext}^2 V$ makes this an isomorphism of Lie algebras with the Clifford algebra[44], allowing the map $\psi : \operatorname{Ext}^2 V \to C(V, Q)$ to be defined by

$$\psi(a \wedge b) = \frac{1}{2}(a \cdot b - b \cdot a) = a \cdot b - Q(a, b) \ . \tag{D.63}$$

This is an injective embedding, which shows that the following Lemma.

Lemma D.3. *The mapping* $\psi \circ \varphi^{-1} : \mathfrak{so}(Q) \to C(Q)^{even}$ *embeds* $\mathfrak{so}(Q)$ *as a Lie subalgebra of* $C(Q)^{even}$.

[44] This is done by checking the brackets on $[a \wedge b, c \wedge d]$ and $[a \cdot b, c \cdot d]$.

Proof. See discussion above. □

The reason why the embedding only goes into the even part is because, simply, $C(Q)^{\text{odd}}$ is indeed not an algebra. You can also see that in Eq. (D.63) the map is defined with elements of even degree. By looking at the basis elements we can then see that ψ is an embedding and the map exactly maps the exterior algebra to the even part of the Clifford algebra.

What remains is to identify the subalgebra of $C(Q)_0$ or $C(Q)^{\text{even}}$, the image of $\mathfrak{so}(Q)$ as matrix algebras. Let us separate this into two cases.

Case 1: $n = \dim V$ is even.

We first decompose V into two n-dimensional isotropic spaces for Q,

$$V = W \oplus W' \ . \tag{D.64}$$

Then we have the following lemma.

Lemma D.4. *The decomposition $V = W \oplus W'$ determines an isomorphism of algebras,*

$$C(Q) \cong \operatorname{End}(\operatorname{Ext}^\bullet W) \tag{D.65}$$

where $\operatorname{Ext}^\bullet W = \operatorname{Ext}^0 W \oplus ... \oplus \operatorname{Ext}^n W$.

Proof. Let us try and construct the map $\varphi : C(Q) \to E = \operatorname{End}(\operatorname{Ext}^\bullet W)$. The map φ is the same as defining a linearly mapping $V \to E$ with the condition in Eq. (D.21). We must therefore construct maps $l : W \to E$ and $l' : W' \to E$ such that

$$l(w)^2 = 0 = l'(w')^2 \tag{D.66}$$

$$l(w) \circ l'(w') + l'(w') \circ l(w) = 2Q(w, w')I \ , \tag{D.67}$$

for any $w \in W$, $w' \in W$. For each $w \in W$, let $L_w \in E$ be the left multiplication by w on the exterior algebra $\operatorname{Ext}^\bullet W$,

$$L_w(\xi) = w \wedge \xi, \quad \xi \in \operatorname{Ext}^\bullet W \ . \tag{D.68}$$

For any $\vartheta \in W^*$, let $D_\vartheta \in E$ be the derivation of $\operatorname{Ext}^\bullet W$ such that,

$$D_\vartheta(1) = 0 \tag{D.69}$$

$$D_\vartheta(w) = \vartheta \in \operatorname{Ext}^0 W = \mathbb{C} \tag{D.70}$$

$$D_\vartheta(\zeta \wedge \xi) = D_\vartheta \zeta \wedge \xi + (-1)^{\deg(\zeta)} \zeta \wedge D_\vartheta(\xi) \ , \tag{D.71}$$

where $w \in W = \operatorname{Ext}^1 W$. i.e. Explicitly,

$$D_\vartheta(w_1 \wedge ... \wedge w_r) = \sum_i (-1)^{i-1} \vartheta(w_i) \left(w_1 \wedge ... \wedge \hat{w}_i \wedge ... \wedge w_r\right) \ . \tag{D.72}$$

Now we can set

$$l(w) = L_w \ , \quad l'(w') = D_\vartheta \ , \tag{D.73}$$

where $\vartheta \in W^*$ is defined by $\vartheta(w) = 2Q(w, w')$, $\quad \forall w \in W$. It is straightforward to show that the maps defined obeys the requirements, as well as for $\zeta \wedge \xi$ if they obey for ζ and ξ separately. The map is clearly an isomorphism and one can see that by its action of a basis. □

Now note that there exists a decomposition of the exterior powers into even and odd parts $\mathrm{Ext}^\bullet W = \mathrm{Ext}^{\mathrm{even}} W \oplus \mathrm{Ext}^{\mathrm{odd}} W$ where $C(W)^{\mathrm{even}}$ respects the splitting. From Lemma D.4, we then have the isomorphism,

$$C(Q)^{\mathrm{even}} \cong \mathrm{End}(\mathrm{Ext}^{\mathrm{even}} W) \oplus \mathrm{End}(\mathrm{Ext}^{\mathrm{odd}} W) \,. \tag{D.74}$$

Combining this with Lemma D.3, we then have an embedding of Lie algebras,

$$\mathfrak{so}(Q) \subset C(Q)^{\mathrm{even}} \cong \mathfrak{gl}(\mathrm{Ext}^{\mathrm{even}} W) \oplus \mathfrak{gl}(\mathrm{Ext}^{\mathrm{odd}} W) \,, \tag{D.75}$$

and we find that there are two representations of $\mathfrak{so}(Q) = \mathfrak{so}_{2n}\mathbb{C}$. We denote the two representations by,

$$S^+ = \mathrm{Ext}^{\mathrm{even}} W \,, \quad S^- = \mathrm{Ext}^{\mathrm{odd}} W \,. \tag{D.76}$$

Proposition D.4. *The representations $S^\pm$ are the irreps of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weights $\alpha = \frac{1}{2}(L_1 + ... + L_n)$ and $\beta = \frac{1}{2}(L_1 + ... + L_{n-1} - L_n)$. More precisely, we have,*

$$S^+ = \Gamma_\alpha \,, \quad S^- = \Gamma_\beta \,, \quad \textit{if } n \textit{ is even}\,; \tag{D.77}$$

$$S^+ = \Gamma_\beta \,, \quad S^- = \Gamma_\alpha \,, \quad \textit{if } n \textit{ is odd}\,. \tag{D.78}$$

Proof. We need to show that the natural basis vectors $e_I = e_{i_1} \wedge ... \wedge e_{i_k}$ for $\mathrm{Ext}^\bullet W$ are weight vectors. Tracing through the isomorphisms, we find that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+i})$ in $\mathrm{Ext}^2 V$, which corresponds to $\frac{1}{2}\left(e_i \cdot e_{n+i} - 1\right)$ in $C(Q)$, and this maps to,

$$\frac{1}{2}\left(L_{e_i} \circ D_{2e_i^*} - I\right) = L_{e_i} \circ D_{e_i^*} - \frac{1}{2} I \in \mathrm{End}(\mathrm{Ext}^\bullet W) \,. \tag{D.79}$$

We can compute,

$$L_{e_i} \circ D_{e_i^*}(e_I) = \begin{cases} e_I & \text{if } i \in I \\ o & \text{if } i \notin I \end{cases} \tag{D.80}$$

. So e_I spans a weight space with weight $\frac{1}{2}\left(\sum_{i\in I} L_i - \sum_{j \notin I} L_j\right)$. All such weights with given $|I| \mod 2$ are congruent by the Weyl group (they are equivalent up to transformations of the Weyl group), so $S^+ = \mathrm{Ext}^{\mathrm{even}} W$ and $S^- = \mathrm{Ext}^{\mathrm{odd}} W$ must be an irreducible representation. The highest weights are then straightforward to read off - the one for $\mathrm{Ext}^{\mathrm{even}} W$ is $\alpha = \frac{1}{2}\sum_i L_i$ if n is even and β if n is odd. □

Definition D.12. The representations $S^\pm$ are the **half-spin representations** of $\mathfrak{so}_{2n}\mathbb{C}$, and $S = S^+ \oplus S^- = \mathrm{Ext}^\bullet W$ is called the **spin representation**. Elements of S are known as **spinors**.

We are going to come back to spinor representations in the next section.

Case 2: $n = \dim V$ **is odd.**

This time we decompose the space V as follows,

$$V = W \oplus W' \oplus U \ , \tag{D.81}$$

where W and W' are n-dimensional isotropic spaces and U is a one-dimensional space perpendicular to them. For the standard Q on $\mathbb{C}^{2n+1}$ these are spanned by the first n, second n, and the last basis vector. We then have the following lemma.

Lemma D.5. *The decomposition* $V = W \oplus W' \oplus U$ *determines an isomorphism of algebras,*

$$C(Q) \cong \mathrm{End}(\mathrm{Ext}^{\bullet} W) \oplus \mathrm{End}(\mathrm{Ext}^{\bullet} W') \ . \tag{D.82}$$

Proof. We can proceed exactly as the even case, as in Lemma D.4. The only difference is with the element u_0 where $Q(u_0, u_0) = 1$. We send u_0 to the endomorphism that is the identity on $\mathrm{Ext}^{\mathrm{even}} W$ and minus the identity on $\mathrm{Ext}^{\mathrm{odd}} W$. This involution then skew commutes with all L_w and D_ϑ, which means the map $V \to E = \mathrm{End}(\mathrm{Ext}^{\bullet} W)$ determines an algebra homomorphism from $C(Q) \to E$. The map for $\mathrm{End}(\mathrm{Ext}^{\bullet} W')$ is similar but with the roles of W and W' reversed. The maps are isomorphic by checking the basis elements. □

From Lemma D.5 we see that the subalgebra $C(Q)^{\mathrm{even}} \subset C(Q)$ ia mapped isomorphically onto the factors,

$$C(Q)^{\mathrm{even}} \cong \mathrm{End}(\mathrm{Ext}^{\bullet} W) \tag{D.83}$$

which gives a representation $S = \mathrm{Ext}^{\bullet} W$ of Lie algebras,

$$\mathfrak{so}_{2n+1}\mathbb{C} = \mathfrak{so}(Q) \subset C(Q)^{\mathrm{even}} \cong \mathfrak{gl}(\mathrm{Ext}^{\bullet} W) = \mathfrak{gl}(S) \ . \tag{D.84}$$

So now we have the following proposition.

Proposition D.5. *The representation* $S = \mathrm{Ext}^{\bullet} W$ *is the irrep of* $\mathfrak{so}_{2n+1}\mathbb{C}$ *with the highest weight*

$$\alpha = \frac{1}{2}(L_1 + ... + L_n) \ . \tag{D.85}$$

Proof. This is similar to the even case - each e_I is an eigenvector with weight $\frac{1}{2}\left(\sum_{i\in I} L_i - \sum_{j\notin I} L_j\right)$. All the weights are congruent by the Weyl group so it must be an irrep with highest weight α. □

We have therefore constructed the spin representations of $\mathfrak{so}_n\mathbb{C}$.

Let us summarise what we have done. We have constructed Clifford algebras via two methods — both abstractly and practically, and then looked at the structure of Clifford algebra itself. We found that the full Clifford algebra consists of the identity $\mathbb{1}$, the D

generating elements Γ_M, and all the independent matrices formed from products of the generators, given by the antisymmetric products,

$$\Gamma_{AB\ldots C} = \Gamma_{[AB\ldots C]} \ . \tag{D.86}$$

Under the canonical automorphism defined on the generators by $\Gamma_M \to -\Gamma_M$, the Clifford algebra inherits a natural $\mathbb{Z}_2$-grading, and it is under this that the Clifford algebra decomposes into even and odd subspaces which consists of real linear combinations of products of an even or odd number of gamma matrices respectively. In particular, the even subspace $C(Q)^{\text{even}}$ is a subalgebra, and we have seen how $\mathfrak{so}(p,q)$ is naturally embedded as a subalgebra in $C(Q)^{\text{even}}$.

D.3 Spinor Representations and Clifford Algebras

Having discussed the spin representations of $\mathfrak{so}_n\mathbb{C}$, it is prudent to discuss its relation with the spinors in this subsection.

D.3.1 Pin and Spin Groups

First, let us define something known as pin and spin groups.

Definition D.13. The **multiplicative group of units** in the Clifford algebra is defined to be the subset

$$C^\times(Q) = \left\{\phi \in C(Q) \,\middle|\, \exists \phi^{-1},\ \phi^{-1}\phi = \phi\phi^{-1} = 1\right\} \ . \tag{D.87}$$

This group contains all elements $v \in V$ with $Q(v) \neq 0$.

The group of units always acts naturally as automorphisms of the algebra, i.e. the **adjoint representation**,

$$\mathrm{Ad} : C^\times(Q) \to \mathrm{Aut}(C(Q)) \ , \tag{D.88}$$

which is given by,

$$\mathrm{Ad}_\phi(x) = \phi x \phi^{-1} \ . \tag{D.89}$$

Taking the derivation of this map gives the usual Lie bracket action $\mathrm{ad}_y(x) = [y, x]$. Hiterto we have assumed that the characteristic of the field could be any integer. Let us assume from now that the chracterisitc of the field $k \neq 2$. Then we have the following important proposition.

Proposition D.6. *Let $v \in V \subset C(Q)$ be an element with $Q(v) \neq 0$. Then $\mathrm{Ad}_v(V) = V$, and $\forall w \in V$, we have,*

$$-\mathrm{Ad}_v(w) = w - 2\frac{Q(v,w)}{Q(v)} v \ . \tag{D.90}$$

Proof. We have that $v^{-1} = -\frac{v}{Q(v)}$, so

$$-Q(v)\,\mathrm{Ad}_v(w) = -Q(v) v w v^{-1} = v w v = -v^2 w - 2Q(v,w)v = Q(v)w - 2Q(v,w)v \ . \tag{D.91}$$

□

Naturally, this lead us to consider the subgroup of elements $\phi \in C^\times(Q)$ such that $\mathrm{Ad}_\phi(V) = V$. From Proposition D.6 above, we see that the group contains all the elements $v \in V$ with $Q(v) \neq 0$, and when this happens the transformation Ad_v preserves the quadratic form Q,

$$(\mathrm{Ad}_v^* Q)(w) = Q(\mathrm{Ad}_v(w)) = Q(w) \ , \tag{D.92}$$

for all $w \in V$. We define $P(Q) \subset C(Q)$ to be the subgroup generated by the elements $v \in V$ with $Q(v) \neq 0$. Note that then there is a representation,

$$P(Q) \to O(V, Q) \ , \tag{D.93}$$

where

$$O(V, Q) = \{\lambda \in GL(V) \,|\, \lambda^* Q = Q\} \tag{D.94}$$

is the orthogonal group of the form Q.

We are now ready to explore the important subgroups of $P(Q)$.

Definition D.14. The **Pin group** is the subgroup of $P(Q)$ generated by the elements $v \in V$ with $Q(v) = \pm 1$, i.e.

$$\mathrm{Pin}(V) = \left\{ v_1 \cdot \ldots \cdot v_m \in C(Q) \,\middle|\, v_j \in S^{n-1} \subset \mathbb{F}^n, m \in \mathbb{N} \right\} \ . \tag{D.95}$$

Definition D.15. The Spin group is similarly defined as,

$$\begin{aligned} \mathrm{Spin}(V) &= \mathrm{Pin}(n) \cap C^0(Q) && \text{(D.96)} \\ &= \left\{ v_1 \cdot \ldots \cdot v_m \in C(Q) \,\middle|\, v_j \in S^{n-1} \subset \mathbb{F}^n, m \in 2\mathbb{N} \right\} \ . && \text{(D.97)} \end{aligned}$$

Note that from the definition of the Pin group, the inverse element to $v_1 \cdot \ldots \cdot v_m$ is,

$$(v_1 \cdot \ldots \cdot v_m)^{-1} = (-v_m) \ldots (-v_1) \in \mathrm{Pin}(V) \ . \tag{D.98}$$

Let us take a deep look at Eq. (D.90). Notice that the right hand side of the equation is basically the reflection $R_v(x)$ of the vector $x \in V$ where v is the vector marking the perpendicular direction of the reflection hyperplane. To remove this sign we therefore consider the following action.

Definition D.16. The **twisted adjoint representation** is the map $\tilde{\mathrm{Ad}} : C^\times(Q) \to GL(C(Q))$ where,

$$\tilde{\mathrm{Ad}}_\phi(y) = \alpha(\phi) y \phi^{-1} \ . \tag{D.99}$$

For even elements ϕ, $\tilde{\mathrm{Ad}}_\phi = \mathrm{Ad}_\phi$. We also have $\tilde{\mathrm{Ad}}_{\phi_1 \phi_2} = \tilde{Ad}_{\phi_1} \circ \tilde{Ad}_{\phi_2}$. Explicitly,

$$\tilde{\mathrm{Ad}}_v(w) = w - 2 \frac{Q(v, w)}{Q(v)} v \ . \tag{D.100}$$

We state without proof the following result.

Theorem D.2 (Cartan-Dieudonné)**.** *Every $g \in O(V)$ is the product of a finite number of reflections $g = R_{u_1} \circ \ldots \circ R_{u_r}$ along the null lines where $Q(u_i) \neq 0$ and $r \leq \dim V$.*

Proof. See [30]. □

We note that the twisted adjoint action must define a group homomorphism $\tilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to O(V)$. It follows from the Cartan-Diedonné Theorem D.2 that $\tilde{\mathrm{Ad}}$ is surjective. But what is the kernel of $\tilde{\mathrm{Ad}}$?

Proposition D.7. *Suppose V is finite dimensional and Q is non-degenerate. Then the kernel of the homorphism $\tilde{\mathrm{Ad}} : \tilde{P}(V,Q) \to GL(V)$ is the group $\mathbb{K}^\times$ of non-zero multiples of 1. Here the group $\tilde{P}(V,Q)$ is defined as,*

$$\tilde{P}(V,Q) = \left\{ \phi \in C^\times(V,Q) \,\middle|\, \tilde{\mathrm{Ad}}_\phi(V) = V \right\} , \tag{D.101}$$

where $P(V,Q) \subset \tilde{P}(V,Q)$.

Proof. See [10] for a complete proof. The proof is also outlined in [11]. □

[10] goes into a bit more detail in how you would define the homomorphism from the group $\tilde{P}(V,Q)$ to the orthogonal group $O(V)$ (see Propositions 2.5 and Corollary 2.6 of [10]). It also shows how the images $\tilde{\mathrm{Ad}}(\mathrm{Pin}(V,Q))$ and $\tilde{\mathrm{Ad}}(\mathrm{Spin}(V,Q))$ is a normal subgroup of $O(V)$ (see Proposition 2.8 of [10]). To summarise there are two exact short sequences.

Theorem D.3. *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and Q a non-degenerate quadratic form on V. Suppose the field $\mathbb{K}$ of characteristic $\neq 2$ is **spin**, i.e. at least one of the two equations $t^2 = a$ and $t^2 = -a$ can be solved in $\mathbb{K}$ for each non-zero element $a \in \mathbb{K}^\times$. Then there are two short exact sequences.*

$$0 \to F \to Spin(V,Q) \xrightarrow{\tilde{\mathrm{Ad}}} SO(V) \to 1 , \tag{D.102}$$

$$0 \to F \to Pin(V,Q) \xrightarrow{\tilde{\mathrm{Ad}}} O(V) \to 1 , \tag{D.103}$$

where

$$F = \begin{cases} \mathbb{Z}_2 = \{1,-1\} & \text{if } \sqrt{-1} \notin \mathbb{K} \\ \mathbb{Z}_4 = \{\pm 1, \pm\sqrt{-1}\} & \text{otherwise} \end{cases} \tag{D.104}$$

The sequences above hold for general fields provided that $SO(V)$ and $O(V)$ are replaced by appropriate normal subgroups of $O(V)$ (since the map $\tilde{\mathrm{Ad}}$ maps to normal subgroups of $O(V)$ in general and field $\mathbb{K}$, which is spin, solves the equation $t^2 Q(v) = \pm 1$ so every $v \in V^\times$ can be renormalised to have $Q(v) = 1$).

Proof. See [10] Theorem 2.9. The details of the field being **spin** is not relevant if we restrict to $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ as both fields are spin. □

The real case of the above Theorem D.3 is reduced to the simple short exact sequences,

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_{r,s} \to SO_{r,s} \to 1 , \tag{D.105}$$

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_{r,s} \to SO_{r,s} \to 1 , \tag{D.106}$$

for all (r, s), where the subscripts denote the signature of the quadratic form Q. In particular, for $SO_n = SO_{n,0} = SO_{0,n}$, we have

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_n \xrightarrow{\tilde{\mathrm{Ad}}} SO_n \to 1 \ , \tag{D.107}$$

where the map $\tilde{\mathrm{Ad}}$ acts as the universal covering map of SO_n for all $n \geq 3$. From this we see that the Pin and Spin groups are covering groups of the orthogonal group. We will now examine the representations of the Pin and Spin groups, which in term will give representations of the orthogonal groups. We will see how this unifies with the picture we have taken in two subsections ago.

D.3.2 Classification of Clifford algebras

Let me say something about the classification of Clifford algebras. This turns out to be very useful in characterising the presence of spinor representations in different dimensions. From this point on we will switch notations from using $C(Q)$ to $\mathrm{Cl}(s, t)$, where the quadratic form Q has signature s and t. We will also define the complexified Clifford algebras as,

$$\mathbb{C}\mathrm{l}(s+t) = \mathrm{Cl}(s,t) \otimes_{\mathbb{R}} \mathbb{C} \ . \tag{D.108}$$

With this we can build up a chessboard of Clifford algebras. Firstly, the low-dimensional Clifford algebras can be explicitly checked.

Lemma D.6. *The low-dimensional Clifford algebras are given by,*

$$Cl(1,0) \cong \mathbb{C} \ , \quad Cl(0,1) \cong \mathbb{R} \oplus \mathbb{R} \ . \tag{D.109}$$

$$Cl(2,0) \cong \mathbb{H} \ , \quad Cl(1,1) \cong Mat_2(\mathbb{R}) \cong Cl(0,2) \ . \tag{D.110}$$

Proof. Explicit computation. □

Then there are the periodicity conditions.

Theorem D.4. *There are isomorphisms (step isomorphisms),*

$$Cl(n,0) \otimes Cl(0,2) \cong Cl(0,n+2) \ , \tag{D.111}$$

$$Cl(0,n) \otimes Cl(2,0) \cong Cl(n+2,0) \ , \tag{D.112}$$

$$Cl(n,m) \otimes Cl(1,1) \cong Cl(n+1,m+1) \ . \tag{D.113}$$

Proof. Proof of these are basically of the following format — take the gamma matrices Γ' from $\mathrm{Cl}(n, 0)$ and Γ'' from $\mathrm{Cl}(0, 2)$ respectively and construct a new basis,

$$\Gamma_a = \begin{cases} \Gamma'_a \otimes \Gamma''_1 \Gamma''_2 & \text{for } 1 \leq a \leq d \ , \\ \mathbb{1} \otimes \Gamma''_{a-d} & \text{for } a = d+1, \ d+2 \ . \end{cases} \tag{D.114}$$

Then we can show that Γ_a are the matrices for $\mathrm{Cl}(0, n + 2)$. The detailed proof is in [10, 26]. □

We will also need the elementary algebraic facts.

Proposition D.8. *The following hold true,*

1. $Mat_n(\mathbb{R}) \otimes Mat_m(\mathbb{R}) \cong Mat_{nm}(\mathbb{R})$.
2. $Mat_n(\mathbb{R}) \otimes_{\mathbb{K}} \mathbb{K} \cong Mat_n(\mathbb{K})$.
3. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.
4. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong Mat_2(\mathbb{C})$.
5. $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong Mat_4(\mathbb{R})$.

Proof. See Proposition 4.2 of [10]. □

Theorem D.5. *We have additional periodicity conditions.*

$$Cl(n+8,0) \cong Cl(n,0) \otimes Cl(8,0) \ , \tag{D.115}$$

$$Cl(0,n+8) \cong Cl(0,n) \otimes Cl(0,8) \ , \tag{D.116}$$

$$\mathbb{C}l(n+2) \cong \mathbb{C}l(n) \otimes_{\mathbb{C}} \mathbb{C}l(2) \ , \tag{D.117}$$

with

$$Cl(0,8) = Cl(8,0) = \mathbb{R}(16) \ , \quad \mathbb{C}l(2) = Mat_2(\mathbb{C}) \ . \tag{D.118}$$

The table obtained is as follows. This table can be further expanded by using the step

n	$\mathrm{Cl}(n,0)$	$\mathrm{Cl}(0,n)$	$\mathbb{C}l(n)$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}$	$\mathrm{Mat}_2(\mathbb{R})$	$\mathrm{Mat}_2(\mathbb{C})$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathrm{Mat}_2(\mathbb{C})$	$\mathrm{Mat}_2(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C})$
4	$\mathrm{Mat}_2(\mathbb{H})$	$\mathrm{Mat}_2(\mathbb{H})$	$\mathrm{Mat}_4(\mathbb{C})$
5	$\mathrm{Mat}_4(\mathbb{C})$	$\mathrm{Mat}_2(\mathbb{H}) \oplus \mathrm{Mat}_2(\mathbb{H})$	$\mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_4(\mathbb{C})$
6	$\mathrm{Mat}_8(\mathbb{R})$	$\mathrm{Mat}_4(\mathbb{H})$	$\mathrm{Mat}_8(\mathbb{C})$
7	$\mathrm{Mat}_8(\mathbb{R}) \oplus \mathrm{Mat}_8(\mathbb{R})$	$\mathrm{Mat}_8(\mathbb{C})$	$\mathrm{Mat}_8(\mathbb{C}) \oplus \mathrm{Mat}_8(\mathbb{C})$
8	$\mathrm{Mat}_{16}(\mathbb{R})$	$\mathrm{Mat}_{16}(\mathbb{R})$	$\mathrm{Mat}_{16}(\mathbb{C})$

Table D.1: Table for classification of Clifford algebras.

isomorphisms into different $\mathrm{Cl}(s,t)$. I won't do it here, but you should have a look at [10] for the nice table.

D.3.3 Spinor Representations

Before we begin let us recall what representations of an algebra is.

Definition D.17. Suppose A is an associative algebra and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. A **$\mathbb{K}$-representation** of A is a $\mathbb{K}$-linear homomorphism $\rho : A \to \mathrm{End}_{\mathbb{K}}(E)$ for some $\mathbb{K}$-vector space E. Two $\mathbb{K}$-representations $\rho : A \to \mathrm{End}_{\mathbb{K}}(E)$ and $\rho' : A \to \mathrm{End}_{\mathbb{K}}(E')$ are **equivalent** if there exists a $\mathbb{K}$-linear isomorphism $f : E \to E'$ such that the following triangle commutes:

$$\begin{array}{ccc} & A & \\ {}^{\rho}\swarrow & & \searrow^{\rho'} \\ \mathrm{End}_{\mathbb{K}}(E) & \xrightarrow{\ \mathrm{Ad}\, f\ } & \mathrm{End}_{\mathbb{K}}(E') \end{array}$$

where $\mathrm{Ad}\, f : \mathrm{End}_{\mathbb{K}}(E) \to \mathrm{End}_{\mathbb{K}}(E')$ is defined as $\phi \mapsto f \circ \phi \circ f^{-1}$, so $f \circ \rho(a) = \rho'(a) \circ f$ for all $a \in A$.

We can now define the following representations.

Definition D.18. A **pinor representation** of $\mathrm{Pin}(V)$ is the restriction of an irreducible representation of $C(Q)$. Similarly, a **spinor representation** of $\mathrm{Spin}(V)$ is the restriction of an irreducible representation of $C_0(Q)$.

It is actually good to stop and define what representations we are exactly shooting for. In physics we encounter four types of spinors — Weyl, Dirac, Majorana and Weyl-Majorana spinors. Let us first give a proper definition, and then we will see how our classification of Clifford algebras above actually give us the full classification of spinor representations in each signature.

Definition D.19. Suppose $\mathrm{Cl}(n)$ is the Clifford algebra of $\mathbb{R}^n$ with its complexification given by $\mathbb{C}\mathrm{l}(n) = \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathbb{C}$. For the standard basis e_i of $\mathbb{R}^n$, define

$$z_j = \frac{1}{2}(e_{2j-1} - ie_{2j}) \in \mathbb{C}\mathrm{l}_n \tag{D.119}$$

for $j = 1, ..., m$, $n = 2m$ and $\bar{z}_j$ be its conjugate. Then the span

$$\Sigma = \{z_{j_1} \cdot ... z_{j_k} \cdot \bar{z}_1 \cdot ... \cdot \bar{z}_m \,|\, k = 0, ..., m, \quad 1 \leq j_1 \leq ... \leq j_k \leq m\} \ , \tag{D.120}$$

defines a complex vector subspace of $\mathbb{C}\mathrm{l}_n$ of dimension 2^m. This space that is known as the **spinor space** with its elements being **spinors**. The spinor space is invariant under Clifford multiplications (multiplication by e_j). We further define $\Sigma^{\pm}$ to be the spans where k is even and odd respectively, and k is known as the **chirality** of the spinor space.

We see how the spinor representation is exactly $E = \Sigma$ in the Definition D.17. Note very carefully that the dimensions of the spinor space Σ has nothing to do with the dimension of V which is the vector space where we have defined the Clifford algebra. This is important for the classification of spinors. Most representations of Clifford algebras (similar to Lie algebras) are reducible. The volume element plays a key role in determining irreducible representations. In particular, since the volume element is used in determining the classification of real and complex Clifford algebras [10], one can use it to determine the properties of irreducible real, complex and quaternionic representations of Clifford algebra. You can read about the details of the classification and determination of irreducible representations in [10] and [11] — I am simply going to make some summary statements here.

We start with complex Clifford algebras (so we can take complex linear combinations of products of gamma matrices). The reason for doing that is that the structure of complexified algebras typically gives uniformity in their representation theory (think Lie algebras [27]!). Then we characterise their representations as follows.

Definition D.20. A **Dirac spinor** is the fundamental complex spinor representation of $\mathrm{Cl}_0(s,t)$, or the map $\Delta_n^{\mathbb{C}} : \mathrm{Spin}(n) \to GL_{\mathbb{C}}(\Sigma)$. This representation is irreducible for odd n.

In even dimensions something specific happens and we have the following kind of spinors.

Definition D.21. A **Weyl spinor** is a complex, irreducible representation of $\mathrm{Cl}_0(s,t)$ in even dimensions.

Okay. How do we understand this categorisation? Recall that to understand spinor representations, the chain of embedding $\mathrm{Spin}(s,t) \subset \mathrm{Cl}_0(s,t) \subset \mathbb{Cl}_0(s+t)$ requires us to look at even subalgebras. There is a fundamental isomorphism given by the following.

Proposition D.9. *There is an algebra isomorphism between even subalgebras of Clifford algebra and one of a higher dimension.*

$$Cl(s,t) \cong Cl_0(s+1,t) \ , \quad \text{for } s \geq 1 \ , \tag{D.121}$$

$$Cl(n) \cong Cl_0(n+1) \ . \tag{D.122}$$

Complexification therefore gives,

$$\mathbb{Cl}(n) \cong \mathbb{Cl}_0(n+1) \ . \tag{D.123}$$

Proof. See Theorem 3.7 of [10]. □

This gives the following categorisation in Table D.2. From this table, we see that in

$d \mod 2$	$\mathbb{Cl}_0(d)$	N
0	$\mathrm{Mat}_N(\mathbb{C}) \oplus \mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$
1	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$

Table D.2: Even subalgebras of a complex Clifford algebra. N is the dimension of the algebra, not the underlying vector space d.

even dimensions d there are two spinor representations of dimension $2^{(d-2)/2}$. These are distinguished by the volume element,

$$\Gamma_{d+1} = \alpha \Gamma_1 \Gamma_2 \dots \Gamma_d \ , \tag{D.124}$$

with $\alpha \in \mathbb{C}$ such that $\Gamma_{d+1}^2 = \mathbb{1}$. Typically $\alpha = (-i)^{\frac{d}{2}+1}$ is picked [9]. This Γ_{d+1}, sometimes denoted as Γ_*, is the highest rank Clifford algebra element, and it (anti-)commutes with all (odd) even rank elements of the algebra,

$$[\Gamma_{d+1}, \mathrm{Cl}_0(d)] = 0 \,. \tag{D.125}$$

This is sometimes known as the **chirality operator**, also defined as the image of the volume element in the Clifford algebra in the representation. From this we can define the projection operators in even dimensions,

$$P_{L,R} = \frac{1}{2}(\mathbb{1} \pm \Gamma_{d+1}) \,, \tag{D.126}$$

and hence project the representation space (or a Clifford module $\Delta_n^{\mathbb{C}}$) to the two eigenspaces of Γ_{d+1}, with eigenvalues $\pm$. This distinguishes the two irreducible representations as *Weyl spinors*. Our argument can be summarised in the following proposition.

Proposition D.10. *Suppose the complex spinor representation is $\Delta_n^{\mathbb{C}} : Spin_n \to GL_{\mathbb{C}}(\Sigma)$ which is given by restricting an irreducible complex representation $\mathbb{Cl}_n \to \mathrm{Hom}_{\mathbb{C}}(\Sigma, \Sigma)$ to $Spin_n \subset Cl_n^0 \subset \mathbb{Cl}_n$. Then for n add this definition is independent of which irrep of $\mathbb{Cl}_n$ is used, and that the representation $\rho_n^{\mathbb{C}}$ is irreducible. When n is even then there is a decomposition,*

$$\Delta_n^{\mathbb{C}} = \Delta_n^{\mathbb{C}^+} \oplus \Delta_n^{\mathbb{C}^-} \,, \tag{D.127}$$

i.e. into a direct sum of two inequivalent irreducible complex representations of $Spin_n$.

Proof. See Proposition 5.15 of [10], which uses the argument for the real case. For more information one can also look at [31]. □

Let us move on to real representations. There are two approaches here — the traditional one starts with the complexified form of the Clifford algebras and look at their representations that comes with a real structure. In this sense, the definition of the spinors is as follows.

Definition D.22. Let S be a spinor representation of a complexified Clifford algebra. We say that S is **Majorana** if S admits a real structure $\mathcal{J}$. A spinor $\psi \in S$ is **Majorana** if it satisfies $\mathcal{J}(\psi) = \psi$.

Definition D.23. Let $S \times V$ be a spinor with flavour representation of a complexified Clifford algebra and some group (with V some quaternionic representation of G), $\mathrm{Spin}(s,t) \times G$. We say that $S \times V$ is **symplectic Majorana** if $S \times V$ admits a quaternionic structure $\mathcal{J}_\otimes$. A spinor $\psi \in S$ is **symplectic Majorana** if it satisfies $\mathcal{J}_\otimes(\psi) = \psi$.

Definition D.24. In the even dimensions where a real structure $\mathcal{J}$ and the chirality projection operator Γ_{d+1} both exists, a spinor is **Majorana-Weyl** if

$$\mathcal{J}(\psi) = \psi \,, \quad \Gamma_{d+1}\psi = \pm i\psi \,. \tag{D.128}$$

Similarly, if $\mathcal{J}$ is a quaternion structure then the spinor that satisfies Eq. (D.128) is called **Majorana-symplectic-Weyl**.

Let me explain how we should interpret this. Firstly, we know that a complex representation that admits a real or quaternionic structure is equivalent to an invariant non-degenerate complex bilinear form B satisfying,

$$B(\Gamma_a \cdot \psi_1, \psi_2) = \tau B(\psi_1, \Gamma_a \cdot \psi_2) \ . \tag{D.129}$$

Here τ is a sign and we also have

$$B(\psi_1, \psi_2) = \epsilon B(\psi_2, \psi_1) \ , \tag{D.130}$$

with $\epsilon = \pm 1$ depending on whether we encode a real or quaternionic structure. We can choose a matrix that represents B by (apologies for using the same notation),

$$\Gamma_a^* = \tau B \cdot \Gamma_a \cdot B^{-1} \ . \tag{D.131}$$

In the classification in [9], the $\tau = -t_0 t_1$, and with

$$B^* B = -t_1 \mathbb{1} \ , \tag{D.132}$$

we can identify $\epsilon = -t_1$. This complex bilinear form gives the action of the real (quaternionic) structure $\mathcal{J} : S \to S$ via the charge conjugate map:

$$\mathcal{J} : \psi \mapsto B^{-1} \psi^* \ . \tag{D.133}$$

Now there is also a *charge conjugation matrix* C which is related to B in [9] via $B = it_0 C \Gamma_0$. Alternatively, given A the volume form (related to the non-degenerate Hermitian form),

$$\Gamma_a = -(-1)^t A \cdot \Gamma_a \cdot A^{-1} \ , \tag{D.134}$$

then,

$$C = t_0 B^T \cdot A \ . \tag{D.135}$$

The reason why C is useful is because we have,

$$\Gamma_a^T = t_0 t_1 C \cdot \Gamma_0 \cdot C^{-1} \ , \quad C^\dagger \cdot C = 1 \ , \tag{D.136}$$

$$C^T = -t_0 C \ . \tag{D.137}$$

What does this imply? we see then $C\Gamma^{(r)}$ is either symmetric or antisymmetric,

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)} \ , \tag{D.138}$$

and we can now look at the possibility of different combinations of t_0 and t_1 that occurs in various dimensions. It turns out the following statements are true:

1. $t_1 = -1$ must hold true for the reality condition to hold, so investigate the cases. $t_0 = +1$ is only possible when $d \mod 8 = 2, 3, 4$. These are the **Majorana spinors**. For $t_0 = -1$ which works for $d \mod 8 = 4, 5, 6$ the spinors are known as **pseudo-Majorana spinors**. There are no real representations, the Clifford algebra generating gamma-matrices instead imaginary.

$s-t \mod 8$	$\mathrm{Cl}_0(s,t)$	N
$1,7$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$
$3,5$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
$2,6$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$
4	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-4)/2}$
0	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-2)/2}$

Table D.3: Structure of the even subalgebras of a real Clifford algebra.

2. For $t_1 = 1$ we cannot define Majorana spinors, but must instead define **symplectic Majorana spinors** which satisfy the condition,

$$\chi^i = \epsilon^{ij} B^{-1} (\chi^j)^* \ , \quad \epsilon^{ij} = -\epsilon^{ji} \ . \tag{D.139}$$

These are exactly the spinors with quaternionic structure above, the weird ϵ^{ij} encodes the quaternionic representation V.

3. Finally in $d \mod 8 = 2$, 4, we see that the Weyl and Majorana conditions are compatible so we have **Majorana-Weyl spinors**. For $d = 0 \mod 4$ dimensions we have,

$$(P_L \psi)^C = P_R \psi \ , \tag{D.140}$$

so the Majorana condition cannot be satisfied.

A lot of this discussion can be simplified if we instead discuss directly the representation content of the real Clifford algebras. There we have Table D.3, From this table you can immediately read out the availabilities of spinor representations.

- For odd dimensions there is a unique spinor representation for $s - t = 1, 7 \mod 8$ — they are **Majorana spinors**.
- For odd dimensions there is a unique spinor representation for $s - t = 3, 5 \mod 8$ — they are **symplectic Majorana spinors**.
- For even dimensions we have two inequivalent representations.
- For $s-t = 2, 6 \mod 8$ there are two inequivalent complex representations (remember the table is before complexification), labelled by chirality — they are **Weyl spinors**.
- For $s - t = 0 \mod 8$ there are two inequivalent complex representations compatible with the real structure — they are **Majorana-Weyl spinors**.
- For $s - t = 4 \mod 8$ there are two inequivalent quaternionic spinor representations — they are **symplectic Majorana-Weyl spinors**.

Note that in the case where $s - t \mod 8 = 2$, i.e. in the case where we have 4d Lorentzian signature, the matrix algebra is $\mathrm{Mat}_N(\mathbb{C})$. The table only gives us the structure of the even subalgebra of the real Clifford algebra, so in this case we know that there is a natural complex module. This however does not rule out a real module (i.e. a Majorana spinor)! Majorana spinors are, at the end of the day, real representations of the Clifford algebra, so we will need to characterise the spinor representations by the real structure.

There is a lot more with the technology of Clifford algebras. You can read [10, 11, 26, 32–34] when you have the time, but I digressed.

E Superspace in detail

Under construction

Working in progress — making this into a better discussion.

E.1 The four-dimensional $\mathcal{N} = 1$ superspace

Recall the Minkowski spacetime M^n is the affine space of the underlying vector space V of translations with Lorentzian metric, and that the Poincaré group P^n is a metric-preserving cover of the component of affine symmetries of M^n connected to the identity. To define a super-spacetime, we will fix a real spin representation S with dimensions s, which has the symmetric pairing [45],

$$\tilde{\Gamma} : S \otimes S \to V \ . \tag{E.1}$$

The related pairing [46] $\Gamma : S^* \otimes S^* \to V$ is used in the supersymmetry algebra. We choose $\mathrm{Im}(\Gamma), \mathrm{Im}(\tilde{\Gamma}) \subset \overline{C}$ where $C \subset V$ is the positive cone of timelike vectors, and then note that Γ and $\tilde{\Gamma}$ are non-degenerate. Choosing a basis of V and S as $\{P_\mu\}$ and $\{Q^a\}$ respectively, we have,

$$\Gamma(Q_a, Q_b) = \Gamma^\mu_{ab} P_\mu \ . \tag{E.2}$$

The Clifford relation between Γ and $\tilde{\Gamma}$ is expressed by,

$$\Gamma^\mu_{ab}\tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab}\tilde{\Gamma}^{\mu bc} = 2g^{\mu\nu}\delta^c_a \ . \tag{E.3}$$

We introduce the $\mathbb{Z}_2$-graded algebra,

$$\mathcal{L} = V \oplus S^* \ , \tag{E.4}$$

with V central and the nontrivial odd bracket,

$$\{Q_a, Q_b\} = -2\Gamma^\mu_{ab} P_\mu \ . \tag{E.5}$$

The underlying supermanifold (affine space) for the corresponding super Lie group is then the **super-Minkowski spacetime**,

$$M^{n|s} = M^n \times \Pi S^* \ , \tag{E.6}$$

[45] The existence of such a symmetric equivariant pairing for real spin representations is unique to the Minkowski signature, see Notes on Spinors in [33].

[46] The Γ is uniquely determined by $\tilde{\Gamma}$ in Lorentzian signature spin representations.

where Π denotes the parity-reversed vector space with the even and odd summands reversed [47].

We typically pick the coordinate basis on V and ΠS^* as x^μ and θ^a respectively, giving the global coordinates on $M^{n|s}$. The action of the Lie algebra $\mathcal{L}$ on $M^{n|s}$ gives rise to left-invariant and right-invariant vector fields with basis $\{\partial_\mu, D_a\}$ and $\{\partial_\mu, \tau_{Q_a}\}$ respectively [48]. We write,

$$D_a = \frac{\partial}{\partial\theta^a} - \Gamma^\mu_{ab}\theta^b\partial_\mu \tag{E.7}$$

$$\tau_{Q_a} = \frac{\partial}{\partial\theta^a} + \Gamma^\mu_{ab}\theta^b\partial_\mu \tag{E.8}$$

with non-trivial brackets [49]

$$[D_a, D_b] = -2\Gamma^\mu_{ab}\partial_\mu \ , \tag{E.9}$$

$$[\tau_{Q_a}, \tau_{Q_b}] = 2\Gamma^\mu_{ab}\partial_\mu \ . \tag{E.10}$$

$$\tag{E.11}$$

Note that since right-invariant vector fields give rise to left actions [50], τ_{Q_a} generates an infinitesimal left action of $P^{n|s}$.

The **super Poincaré** algebra is the graded Lie algebra,

$$\mathfrak{p}^{n|s} = (V \oplus \mathfrak{so}(V)) \oplus S^* \ , \tag{E.12}$$

its even part just the usual Poincaré algebra. The **super Poincaré group** is defined as $P^{n|s} = \mathrm{Spin}(V) \ltimes \exp(\mathcal{L})$ or equivalently,

$$1 \to \exp(\mathcal{L}) \to P^{n|s} \to \mathrm{Spin}(V) \to 1 \ . \tag{E.13}$$

There are two concepts connected to $P^{n|s}$.

1. There may be a symmetric pairing

$$S^* \otimes S^* \to \mathbb{R}^c \ , \tag{E.14}$$

i.e. $\mathrm{Sym}^2 S^*$ contains copies of the trivial representation. This leads to the extension of a new-super Lie algebra by adding for any $c' \le c$,

$$\tilde{\mathfrak{p}}^{n|s} = \left(V \oplus \mathfrak{so}(V) \oplus \mathbb{R}^{c'}\right) \oplus S^* \ . \tag{E.15}$$

[47] This is consistent with the sign rule where we introduce a sign when odd elements are interchanged and treat all structures as even.

[48] ∂_μ in both as V is central.

[49] The brackets of the left-invariant D_a are given by the ones as in $\mathcal{L}$ but the right-invariant ones have a sign difference.

[50] This is because when the one-parameter subgroup generated by a vector field v acts on the left, say $g \mapsto e^{tv} \cdot g$, after differentiation g acts on the right of the vector field as

$$\left.\frac{dg'}{dt}\right|_{t=0} = ve^{tv}g = v \cdot g$$

so the vector field generated by v gives vg at $g \in G$. This means that the vector field is in fact right-invariant.

Since we can write

$$1 \to \mathbb{R}^{c'} \to \tilde{\mathfrak{p}}^{n|s} \to \mathfrak{p}^{n|s} \to 1 \ , \tag{E.16}$$

this is an extension and in particular $\mathbb{R}^{c'}$ is abelian with its image being in the centre of $\tilde{\mathfrak{p}}^{n|s}$ so such construction is known as a **central extension**. The generators of $\mathbb{R}^{c'}$ are the **central charges** [51].

2. There may exists outer automorphisms of $\mathfrak{p}^{n|s}$ which fix the Poincaré algebra (so they transform the fermionic generators). These are **infinitesimal R-symmetries** and the connected group via exponentiation is the R-symmetry group, which is compact [52].

E.2 Supermanifolds and superspace

In fact supermanifolds can be constructed in an abstract manner. We will discuss the construction of supermanifolds from an algebraic point of view. To do this let us first focus on the simplest case.

Superspacetime of particles

Suppose $W^0 = \mathbb{R}^m$ to be space. Then the fields in this theory that describe particles is the embedding,

$$\mathcal{F}^0 = \{x : \mathbb{R} \to W^0\} \ . \tag{E.17}$$

For a free non-relativistic particle, we have the classical equation of motion $\ddot{x}(t) = 0$. With the embedding $\mathcal{M}^0 \to W^0 \oplus W^0$ where $x \mapsto x(t_0) \oplus \dot{x}(t_0)$, we can define a symplectic structure,

$$\omega = m\langle \delta\dot{x}(t_0) \wedge \delta x(t_0)\rangle \ , \tag{E.18}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on W^0 and δ is the differential on $\mathcal{M}$ [53].

A problem arises when we replace $\mathcal{M}^0$ by a graded vector space, the odd symplectic vector space $\mathcal{M}^1$ now gives a symmetric ω and further imposing $\omega(v, v) < 0$ for $v \in \mathcal{M}^1$ still

[51] These already arise in classical field theories due to the symplectic structure. Namely if $\mathfrak{p}^{n|s}$ is the Lie algebra of symmetries of some theory then the Lie algebra of observables is in general a central extension.

[52] The infinitesimal R-symmetries act in a quantum theory as automorphisms of the symmetry algebra and are represented projectively on the Hilbert space of the theory.

[53] The quantisation of this space, which assigns to this symplectic vector space (as the tangent space $W^0 \oplus W^0$) to a complex Hilbert space $\mathcal{H}^0$, can then be proceeded by either demanding that the symmetries of the space be represented as unitrary operators on $\mathcal{H}^0$. This requires a projective representation of the symplectic group which is constructed by the splitting (polarisation) into two Lagrangian subspaces,

$$\mathcal{M}^0 \cong L \oplus L' \ . \tag{E.19}$$

The Hilbert space is the space of L^2-functions on L , which includes the dense subspace of polynomial functions, $\mathcal{H}^0 \supset \mathrm{Sym}^\bullet(L^*) \otimes \mathbb{C}$. There exists a metaplectic representation such that the quantum Hilbert space is the underlying complex Hilbert space of that representation. Alternatively, one can proceed by looking at the Poisson algebra of the functions on the symplecic vector space and assign to each real-valued function on $\mathcal{M}^0$ a self-adjoint operator $\hat{f}$ on $\mathcal{H}^0$.

requires proper isotropic subspaces. To resolve this, we complexify and consider complex polarisations,

$$\mathcal{M}^1 \otimes \mathcal{C} \cong L_{\mathbb{C}} \oplus \overline{L_{\mathbb{C}}} \ , \tag{E.20}$$

so then we have the Hilbert space,

$$\mathcal{H} = \mathrm{Sym}^\bullet(L_{\mathbb{C}}^*) \ . \tag{E.21}$$

This is infact exactly the same as the construction of spin representation of orthogonla groups in even dimensions. The underlying projective space is then canonically independent of polarisation, same as before with the metaplectic representation.

Now it is possible to define the function space of fermionic particles as,

$$\mathcal{F}^1 = \{\psi : \mathbb{R} \to W^1\} \ . \tag{E.22}$$

To describe W^1 as a manifold, we need to specify its space of smooth functions (via the sheaf construction). Recall for an even vector space W^0 the ring of smooth functions contains the dense subset of polynomial functions,

$$C^\infty(W^0) \supset \mathrm{Sym}^\bullet\left((W^0)^*\right) \ , \tag{E.23}$$

so could we perhaps define,

$$C^\infty(W^1) \supset \mathrm{Sym}^\bullet\left((W^1)^*\right) \ , \tag{E.24}$$

as an analogy? The ring of functions $C^\infty(W^1)$ contains a large subring of nilpotent elements,

$$C^\infty(W^1)/\{\text{nilpotents}\} \cong \mathbb{R} \ . \tag{E.25}$$

This is a commutative ring with nilpotents [54]. Geometrically, we should therefore think of W^1 as a point with a nilpotent cloud surrounding it, giving any $\psi : \mathbb{R} \to W^1$ identically

[54] A simple example is the space $P = \mathrm{Spec}\mathbb{R}[\epsilon]/(\epsilon^2)$. A smooth map between manifolds is equivalent to an algebra homomorphism on functions in the opposite direction, so a map $P \to M$ is given by,

$$C^\infty(M) \to \mathbb{R}[\epsilon]/\left(\epsilon^2\right) \ , \tag{E.26}$$

$$f \mapsto A(f) + B(f)\epsilon \ . \tag{E.27}$$

Now the fact that this is an algebra homomorphism leads to two conclusions.

1. $A : C^\infty(M) \to \mathbb{R}$ is algebra homomorphism, so must be the evaluation map

$$A(f) = f(m) \ . \tag{E.28}$$

2. B must be a derivative over functions, so must be of the form,

$$B(f) = \xi_m f \ , \tag{E.29}$$

   with $\xi_m \in T_m$.

This means the maps are in one-to-one correspondence with the tangent bundle of M. The way to picture P is a point together with a cloud of 'nilpotents' surrounding it. Similarly, the maps $M \to P$ must map everything to the geometric point in P.

zero.

To effectively study the maps $\mathbb{R} \to W$, one should introduce nilpotents in the domain by studying families of maps parametrised by a space with niolpotents. Defining B to be a point with a cloud of superfuzz,

$$C^{\infty}(B) = B[\eta_1, ..., \eta_N] \; , \quad \eta_i \eta_j = -\eta_j \eta_i \; , \tag{E.30}$$

then the product space $B \times \mathbb{R}$ gives functions,

$$C^{\infty}(B \times \mathbb{R}) = \mathbb{R}[\eta_1, ..., \eta_N] \otimes C^{\infty}(\mathbb{R}) \; . \tag{E.31}$$

There are non-trivial maps $B \times \mathbb{R} \to W^1$. In particular, for the simplest case $C^{\infty}(B) = \mathbb{R}[\eta]$, we choose a basis $\{\psi^i\}$ of $(W^1)^*$ then,

$$C^{\infty}(W^1) = \mathbb{R}[\psi^1, ..., \psi^m] \tag{E.32}$$

so then the map $\psi : B \times \mathbb{R} \to W^1$ will be given by $C^{\infty}(W^1) \to \mathbb{R}[\eta] \otimes C^{\infty}(\mathbb{R})$ as

$$\psi^i \mapsto a^i(t)\eta \; , \tag{E.33}$$

for some functions $a^i \in C^{\infty}(\mathbb{R})$, which maps now vanish at the geometric point of B ($\eta = 0$).

It is now clear what the Lagrangian density defined by,

$$L^1 = \frac{m}{2} \langle \psi(t), \dot{\psi}(t) \rangle |dt| \; , \tag{E.34}$$

can be represented in this language - we will need maps of the form,

$$\psi^i(t) = a^i(t)\eta_1 + b^i(t)\eta_2 \; , \tag{E.35}$$

which gives,

$$L^1 = \eta_1 \eta_2 \left[\frac{m}{2} g_{ij} \left(a^i(t) \dot{b}^j(t) - b^i(t) \dot{a}^j(t) \right) dt \right] \; , \tag{E.36}$$

which then gives an even action.

The manifolds described by this sheaf-theoretical construction with superfuzz are known as supermanifolds, and it is in these manifolds and spaces that the theory of supersymmetry is developed [34].

Supermanifolds as ringed spaces

Let us provide a robust definition of supermanifolds.

Definition E.1. Let $\mathcal{C}^{\infty}$ be the sheaf of C^{∞}-functions on $\mathbb{R}^m$. The space $\mathbb{R}^{m|n}$ is the topological space $\mathbb{R}^m$ endowed with the sheaf $\mathcal{C}^{\infty}[\theta^1, ..., \theta^n]$ of commutatie super $\mathbb{R}$-algebras freely generated over $\mathcal{C}^{\infty}$ by odd quantities $\theta^1, ..., \theta^n$.

Definition E.2. A **supermanifold** M of dimension $m|n$ is a topological space $|M|$ with a sheaf of super $\mathbb{R}$-algebras which is locally isomorphic to $\mathbb{R}^{m|n}$. The supermanifold **structure sheaf** is denoted as $\mathcal{O}_M$.

We note that the odd functions generate a nilpotent ideal $J \subset \mathcal{O}_M$, and $(|M|, \mathcal{O}_M/J)$ is a C^∞-manifold of dimension m, locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty)$.

Super-Minkowski Space $M^{n|s}$

Before we continue let us define some mathematical notions.

Definition E.3. A **super-Minkowski space** $M^{1,n-1|s}$ contains the following data.

1. A Minkowski space $M^{1,n-1}$ with vector space of translations V.
2. A positive cone C of timelike vectors in V.
3. Areal spinorial representation S of $\mathrm{Spin}(V)$.
4. A symmetric, positive-definite morphism Γ of representations of $\mathrm{Spin}(V)$,

$$\Gamma : S^* \otimes S^* \to V \ . \tag{E.37}$$

such that $\Gamma(s^*, s^*) \in \overline{C}$ with $s^* \in S^*$ and only zero when $s^* = 0$.

There is a unique symmetric morphism,

$$\tilde{\Gamma} : S \otimes S \to V \ , \tag{E.38}$$

Picking a basis $\{e_\mu\}$ and $\{f^1\}$ for V and S allows us to relate Γ and $\tilde{\Gamma}$ as,

$$\Gamma^\mu_{ab}\tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab}\tilde{\Gamma}^{\mu bc} = 2g^{\mu\nu}\delta^c_a \ . \tag{E.39}$$

With the metric g on V it is now possible to convert the morphisms Γ into $\gamma : V \to \mathrm{Hom}(S^*, S)$. Together with $\tilde{\gamma} : V \to \mathrm{Hom}(S, S^*)$, it turns $S \oplus S^*$ into a module over the Clifford algebra $C(V)$, which induces the action of $\mathrm{Spin}(V)$ on S and S^*. Now the uniqueness of Γ up to automorphsims of S allows the definition of an isomrphism $\alpha : S \to S^*$ (if the rep is isomorphic to its dual) such that

$$\tilde{\Gamma}(s, t) = \Gamma(\alpha(s), \alpha(t)) \ , \tag{E.40}$$

which allows a definition of the ϵ as

$$\epsilon(s, t) = \alpha(s)(t) \ . \tag{E.41}$$

The different cases of S in different dimensions is treated in [**?**]. To summarise, there is a dual form $\tilde{\epsilon}$ on S^* which gives in all cases,

$$\epsilon^{ab}\tilde{\epsilon}_{cb} = \delta^a_c \ . \tag{E.42}$$

Note that if the dimension $s = \dim S$ is 2 or 6 $\mod 8$, there are super-Minkowski spaces based on unequal number of copies of half spinor representations S^+ and S^-. Then the invariant pairings Γ and $\tilde{\Gamma}$ is now between S^+ and S^- so they encode information not related by a self-duality pairing [33].

In particular, if we consider the Lie algbera,

$$\exp \mathcal{L} = V \times \Pi S^* \ , \tag{E.43}$$

and fix the basis as before, the nontrivial brackets will be,

$$[f_a, f_b] = -2\Gamma^\mu_{ab} P_\mu \ , \tag{E.44}$$

which is just,

$$[Q_a, Q_b] = -2\Gamma^\mu_{ab} P_\mu \ . \tag{E.45}$$

This basis induces a coordindate system (x^μ, θ^a) on $\exp(\mathcal{L})$, so if $p \in (\mathcal{L} \otimes R)^0$ with R a supercommutative ring, then,

$$p = e_\mu x^\mu + f_a \theta^a \ . \tag{E.46}$$

We write D_a and τ_{Q_a} to be the left-invariant and right-invariant vector fields ∂_a at the origin. Then we have the coordinate representation,

$$D_a = \partial_a - \theta^b \Gamma^\mu_{ab} \partial_\mu \ , \tag{E.47}$$

$$\tau_{Q_a} = \partial_a + \theta^b \Gamma^\mu_{ab} \partial_\mu \ , \tag{E.48}$$

with ∂_μ being both left- and right-invariant. Then we have,

$$[D_a, D_b] = -2\Gamma^\mu_{ab} \partial_\mu \ , \tag{E.49}$$

$$[\tau_{Q_a}, \tau_{Q_b}] = 2\Gamma^\mu_{ab} \partial_\mu \ , \tag{E.50}$$

$$[D_a, \tau_{Q_b}] = 0 \ . \tag{E.51}$$

The last follows from the fact that right and left translations commute.

Supermanifolds and superspaces

Now we ask the question - since $\mathcal{Q}$ is approximately a square root fo the infinitesimal time translation, is it possible that there is a time translation operator whose square-root will generate $\mathcal{Q}$ on the space of fields?

Turns out this requires instead replacing the manifold (and not the target manifold) M by a supermanifold in whihc supersymmetry is manifest. This $M^{m|n}$ is known as superspacetime or more simply, the superspace. In particular, in the case we have been working so far we have considered the following,

$$x, \psi : M^n \to \Pi TX \ , \tag{E.52}$$

where $\Psi \in \Omega^0_{\mathcal{M}^n}(x^* \Pi TX)$, the pullback of the parity-reversed [55] space along the map $x : M^n \to X$. Now we consider, instead, the field,

$$\Phi : M^{m|n} \to X \ . \tag{E.53}$$

[55]Recall the operation Π reverses the parity of the vector space, so $(\Pi W)^0 = W^1$, for example. Alternatively we can think of $\Pi W = W \otimes \mathbb{R}^{\mathrm{odd}}$.

The affine space is now the ringed space of

$$C^\infty(M^{m|n}) = C^\infty(M^m)[\theta^1, ..., \theta^n] \tag{E.54}$$

for odd variables θ^i. We define the two formulations as the **component formalism** and **superfield formalism** respectively.

To recap, we have two formalisms for field theory in supersymmetry.

Definition E.4. Let $M^{1,n-1|s}$ be a super-Minkowski space with real dimensions n (with Poincaré group $\mathcal{P} = V \rtimes \mathrm{Spin}(1, n-1)$ [56], with a real spinorial representations S of $\mathrm{Spin}(V)$ amd a symmetric, positive-definite morphism $\Gamma : S^* \otimes S^* \to V$ of representations of $\mathrm{Spin}(V)$ such that $s = \dim(S)$. The fields on $M^{1,n-1|s}$ are then known as **superfields**, with typical examples,

- Maps $\Phi : M^{1,n-1|s} \to X$ with X an ordinary manifold.
- Set of connections on principal bundle $\mathcal{P} \to M^{1,n-1|s}$.

Field theory formulated using superfields is called the **superfield formalism**.

Definition E.5. Component fields are ordinary even and odd fields on Minkowski space $M^{1,n-1}$ [57]. In particular, they are fields of the form,

$$f : M^{1,n-1} \to X, x^*\Pi TX, \ldots \, . \tag{E.55}$$

Given a superfield Φ, it is possible to define its component fields to be the restriction to $M^{1,n-1}$ of certain derivatives of Φ. We let the inclusion map of Minkowski space be $\iota : M^{1,n-1} \hookrightarrow M^{1,n-1|s}$, then for $\Phi : M^{1,n-1|s} \to X$ we define a multiplet of component fields $\{f^{(r)}\}$ with the following formulae,

$$f^{(r)} = \iota^* D^r \Phi \ , \tag{E.56}$$

with D the left-invariant vector fields on $M^{1,n-1|s}$ [58].

In particular, there is a diffeomorphism $e^{\eta^a \tau_{Q_a}}$ of M and we define $\hat{\eta}$ to be the change under this diffeomorphism,

$$\hat{\eta} f = \frac{D}{Dt}\bigg|_{t=0} \iota^* D^r \big(\big(e^{-t\eta^a \tau_{Q_a}}\big)^* \Phi \big) = -\eta^a \iota^* D_a D^r \Phi \ . \tag{E.57}$$

This allows us to calculate the supersymmetry variations of the component fields.

Finally we discuss how the component Lagrangian in the component formalism works. The idea is to start from L_S on superspace and integral L_S over the odd variables using the splitting $M_S = M \times \Pi S^*$. This determines a projection,

$$\pi : M_S \to M \tag{E.58}$$

[56] There is a positive cone C of timeilke vectors in V, the vector space of translations.
[57] Throughout this note I will denote super-Minkowski spaces with two numbers with a vertical line in between, whereas ordinary Minkowski spaces will be denoted by the metric signature with no vertical lines.
[58] This ensures that the covariant definitions of component fields are global and are patch-indepedent.

from super-Minkowski space M_S to the Minkowski space M. The pushforward gives the integration on densities,

$$\pi_* : \mathrm{Dens}(M_S) \to \mathrm{Dens}(M) \ . \tag{E.59}$$

To illustrate this, start with a Lagrangian function l on M_S,

$$L_S = |d^n x| d^s\theta l \ , \tag{E.60}$$

where $|d^n x| \in \mathrm{Dens}(M)$ and $d^s\theta$ the volume form on S^*. Then the **component Lagrangian** L on M is defined as,

$$|d^n x| L = |d^n x|(\iota^* D^s l) \ , \tag{E.61}$$

this compoenent Lagrangian L function is naturally expressed in terms of component fields. Note that the pushforward of L_S is related to L by a Poincaré invariatn differential operaotr Δ on M, which gives,

$$\Pi_* L_S = |d^n x|(L + \Delta\iota^* l) \ . \tag{E.62}$$

Normally this Δ is a wave operator useful in computations.

E.3 Examples and superfields

E.3.1 Superparticle

Let us first consdider the simplest case of a particle moving in a line. The time coordinate is the affine coordinate t on M^1, giving global coordinates t, θ on $M^{1|1}$. The embedding map is now,

$$\iota : M^1 \hookrightarrow M^{1|1} \ , \tag{E.63}$$

where

$$\iota^* t = t \ , \quad \iota^* \theta = 0 \ . \tag{E.64}$$

Let us simultaneously define the vector fields,

$$\partial_t = \partial/\partial t \tag{E.65}$$

$$\partial_\theta = \partial/\partial\theta \tag{E.66}$$

$$D = \partial_\theta - \theta\partial t \tag{E.67}$$

$$\tau_Q = \partial_\theta + \theta\partial t \tag{E.68}$$

Here ∂_t is even whilst D and τ_θ are odd - $\{\partial_t, D\}$ and $\{\partial_t, \tau_Q\}$ form the left- and right-invariant vector field basis of $M^{1|1}$. We have the bracket relations,

$$[D, D] = -2\partial_t \ , \tag{E.69}$$

$$[\tau_Q, \tau_Q] = 2\partial_t \ . \tag{E.70}$$

We now formulate a field theory with the superfield $\Phi : M^{1|1} \to \mathbb{R}$ with the component fields defined by,

$$x = \iota^* \Phi \ , \tag{E.71}$$

$$\psi = \iota^* D\Phi \ . \tag{E.72}$$

To compute the supersymmetry variations, we must have $\hat{\eta}$ corresponding to $e^{\eta Q}$ in superspace which gives,

$$\hat{\eta}x = -\eta\iota^* D\Phi = -\eta\psi \ , \tag{E.73}$$

$$\hat{\eta}\psi = -\eta\iota^* D^2\Phi = \eta\dot{x} \ , \tag{E.74}$$

which can be deduced by acting τ_Q on the superfield as well [59],

$$(-\eta\tau_Q)\Phi(t,\theta) = -\eta\psi(t) + \theta\eta\dot{x}(t) \ . \tag{E.76}$$

Similarly, bracketing can be checked like,

$$(\hat{\eta}_1\hat{\eta}_2 - \hat{\eta}_2\hat{\eta}_1)x = 2\eta_1\eta_2\dot{x} \ . \tag{E.77}$$

The Lagrangian density for the superfield can be now written as,

$$\mathcal{L}_S = |dt|d\theta \left[-\frac{1}{2}D\Phi\partial_t\Phi\right] \ . \tag{E.78}$$

To compute the component Lagrangian, we integrate out the odd variable as per Eq. (E.61), to get

$$L = dt(\iota^* Dl) \tag{E.79}$$

To evaluate this we compute,

$$\begin{aligned}
\iota^* Dl &= -\frac{1}{2}\iota^* DD\Phi, \partial_t\Phi \\
&= -\frac{1}{2}\iota^*\left[\nabla_D D\Phi, \partial_t\Phi - D\Phi\nabla_D\partial_t\Phi\right] \\
&= -\frac{1}{2}\iota^*\left[-|\partial_t\Phi|^2 - D\Phi\nabla_{\partial_t}D\Phi\right] \\
&= \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\psi\dot{\psi}
\end{aligned} \tag{E.80}$$

which exactly gives the superparticle component Lagrangian - a boson with its fermionic partner moving in a line.

In particular, one can compute the Noether current j associated to the symmetry $\hat{\eta}$. To do this we compute the variational one-form [60],

$$\gamma = \dot{x}\delta x + \frac{1}{2}\psi\delta\psi \ ,. \tag{E.81}$$

[59] One needs to use the odd parameter η in order to fix the signs, otherwise there will be an error incurred. See for example [**?**], where instead of Eq. (E.76) we instead get,

$$(-\tau_Q)\Phi(t,\theta) = -\psi(t) - \theta\dot{x}(t) \ , \tag{E.75}$$

which gives the wrong sign on the ψ transformation.

[60] Recall that the variational one-form $\gamma \in \Omega^{1,|-1|}_{\text{loc}}(\mathcal{F}\times M)$ is defined to be the usual integration-by-parts part when one computes the Euler-Lagrange equations. the total Lagrangian is then $\mathcal{L} = L + \gamma \in \Omega^{|0|}(\mathcal{F}\times M)$, where as the symplectic form is defined to be $\omega = \delta\gamma \in \Omega^{2,|-1|}_{\text{loc}}(\mathcal{F}\times M)$, with d the exterior derivative on M and δ the differential on $\mathcal{F}$.

To compute the current we also need the variation of the Lagrangian under the symmetry [61], which is computed as,

$$\hat{\eta}L = \partial_t \left(-\frac{1}{2}\eta\psi\dot{x} \right) \ , \tag{E.83}$$

so now we can compute the Noether current,

$$j = \iota(\hat{\eta})\gamma - \left(-\frac{1}{2}\eta\psi\dot{x} \right) = -\eta\dot{x}\psi \ , \tag{E.84}$$

which the supercharge is defined to be minus the Noether current,

$$Q = \dot{x}\psi \ . \tag{E.85}$$

E.3.2 $\mathcal{N} = 1$ chiral multiplet on $M^{1,3}$

We now go back to the famililar Minkowski space $M^{1,3}$ and its superspace $M^{1,3|4}$. The superfield is then defined as,

$$\Phi : M^{4,4} \to \mathbb{C} \ . \tag{E.86}$$

The simplest superfield that gives a sensible Lagrangian is the **chiral superfield** which is required to satisfy,

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{E.87}$$

The component fields are defined as,

$$\phi = \iota^*\Phi \ , \tag{E.88}$$

$$\psi_a = \frac{1}{\sqrt{2}}\iota^* D_a\Phi \ , \tag{E.89}$$

$$F = \frac{1}{2}\iota^*(-D^2)\Phi \ . \tag{E.90}$$

Here $\phi : M^4 \to X$, ψ is a spinor field with values in $\Phi^*\Pi TX$, and F is an **auxiliary** field in the sense that for fundamental Lagrangians no derivatives of F occur in the Lagrangian - it is non-dynamical. Note that if X if curved then D needs to be a covariant derivative. For the case where X is linear (flat), we can compute directly the supersymmetric variations of the component fields as before using $\hat{\xi}$, the vector field on the space of component fields induced by the SUSY transformation $\eta^a Q_a + \bar{\eta}^{\dot{a}}\bar{Q}_{\dot{a}}$. The results of the computations are,

$$\hat{\xi}\phi = -\sqrt{2}\eta^a\psi_a \ , \tag{E.91}$$

$$\hat{\xi}\psi_a = \sqrt{2}\left(\bar{\eta}^{\dot{b}}\partial_{a\dot{b}}\phi - \eta^b\epsilon_{ab}F \right) \ , \tag{E.92}$$

$$\hat{\xi}F = \sqrt{2}\bar{\eta}^{\bar{a}}\left(\not{D}\psi \right)_{\dot{a}} \ . \tag{E.93}$$

[61] Recall a local vector field $\hat{\xi}$ on $\mathcal{F}$ is a **generalised infinitesimal symmetry** of L if there exists $\alpha_{\hat{\xi}} \in \Omega^{0,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ such that $\text{Lie}(\hat{\xi})L = d\alpha_{\hat{\xi}}$. Then the Noether current is defined to be,

$$j_\xi = [\iota(\xi)\mathcal{L}]^{0,|-1|} - \alpha_\xi \in \Omega^{0,|-1|}_{\text{loc}}(\mathcal{F} \times M) \ , \tag{E.82}$$

where $\xi = \hat{\xi} + \eta$ with η generating a generalised form of a manifest infinitesimal symmetry ξ.

Now the Lagrangian density on $M^{4|4}$ is,

$$\mathcal{L}_{0,S} = |d^4x|d^2\theta\frac{1}{4}\bar{\Phi}\Phi \ . \tag{E.94}$$

The component Lagrangian is calcualted by acting $\int d^4\theta + \Box\iota^*$ which gives the Lagrangian density on M^4,

$$\mathcal{L}_0 = \eta^{a\bar{b}}\partial_a\phi\partial_{\bar{b}}\bar{\phi} + \bar{\psi}\not{D}\psi + \bar{F}F \tag{E.95}$$

where I have dropped an exact term $-\frac{1}{2}\partial_\mu(\bar{\psi}\Gamma^\mu\psi)$.

It is possible to add in a potential term. First note that if Λ is any chiral superfield then

$$|d^4x|\int d^2\theta\Lambda \tag{E.96}$$

makes sense as a density on Minkowski space (and similarly for $\bar{\Lambda}$ antichiral) [62]. Now if $W : X \to \mathbb{C}$ is a holomorphic function, we can set,

$$\mathcal{L}_{1,S} = |d^4x|\,\mathrm{Re}\left[d^2\theta\Phi^*(W)\right] \ , \tag{E.97}$$

so then,

$$\int \mathcal{L}_{1,S} = \int |d^4x|\frac{1}{2}\left[\int d^2\theta\Phi^*(W) + \int d^2\bar{\theta}\bar{\Phi}^*(\bar{W})\right] = \int |d^4x|\mathcal{L}_1 \ . \tag{E.98}$$

This gives us the extra terms as we seen before,

$$\mathcal{L}_1 = -|\partial_a W|^2 - \frac{1}{2}\left(\partial_a\partial_b W\psi^a\psi^b + \bar{\partial}_{\dot{a}}\bar{\partial}_{\dot{b}}W\bar{\psi}^{\dot{a}}\bar{\psi}^{\dot{b}}\right) \ , \tag{E.99}$$

after eliminating the auxiliary fields,

$$F = -\phi^*\mathrm{grad}W \ , \tag{E.100}$$

$$\bar{F} = -\phi^*\overline{\mathrm{grad}W} \ . \tag{E.101}$$

In particular, putting all the mathematical rigour aside, the action simply reduces to the cases the chiral multiplet action we have in the lectures. This section simply formalises the notions of superspace in more mathematical detail but the notions are physically clear.

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