# Class 4 Topics

- finish up NLSMs.
- Vector Superfields & Non-abelian gauge theories.
- SUSY vacua + moduli spaces
- R-symmetry interlude
- Symmetry-breaking.

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## Vector Superfields

- Vector fields are real. Want to impose a reality condition.

$$V = \bar{V}$$

This gives

$$V(x,\theta,\bar\theta) = C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \theta\sigma^\mu\bar\theta v_\mu + \frac{i}{2}\theta\theta(M+iN) - \frac{i}{2}\bar\theta\bar\theta(M-iN)$$
$$+ i\theta\theta\bar\theta\left(\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi\right) - i\bar\theta\bar\theta\theta\left(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi\right) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(D - \frac{1}{2}\partial^2 C\right)$$

$$8_F + 8_B \xrightarrow[\text{gauge fix}]{} 4_F + 4_B \xrightarrow[\text{on shell}]{} 2_F + 2_B$$

If we have

$$V \mapsto V + \Phi + \bar\Phi$$
$$v_\mu \mapsto v_\mu - \partial_\mu(2\,\mathrm{im}\,\phi) \quad \text{abelian!}$$

The transf

$$C \to C + 2\mathrm{Re}\,\phi$$
$$\chi \to \chi - i\sqrt{2}\psi$$
$$M \to M - 2\,\mathrm{im}\,F$$
$$N \to N + 2\,\mathrm{re}\,F$$

$D, \lambda$ inv

$v^\mu$ above.

Pick gauge:

$$V_{WZ} = \theta\sigma^\mu\bar\theta v_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x)$$

1 $\theta$ at least!

$$V_{WZ}^2 = \frac{1}{2}\theta\theta\bar\theta\bar\theta\, v_\mu v^\mu.$$

We need a field strength.

$$\begin{cases} W_\alpha = -\frac{1}{4}\bar D\bar D D_\alpha V \\ \bar W_{\dot\alpha} = -\frac{1}{4}DD\bar D_{\dot\alpha} V. \end{cases}$$

$W_\alpha$ chiral since

$$\bar D W = 0.$$

Invariant:

$$W_\alpha \longmapsto W_\alpha - \frac{1}{4}\bar{D}\bar{D}D_\alpha(\Phi + \bar{\Phi})$$

$$\bar{D}^{\dot{\beta}}\{\bar{D}_{\dot{\beta}}, D_\alpha\} \sim \sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \bar{D}^{\dot{\beta}}$$

WZ gauge:

gaugino.

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha$$

SUSY field strength.

F-term?

$$\int d^2\theta\, W^\alpha W_\alpha = -\frac{1}{2}F^2 - 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 + \frac{i}{4}F\wedge F.$$

$+$ c.c.

$$= -F^2 - 4i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + 2D^2.$$

Look, not F-term

$$\int d^2\theta\, W^\alpha W_\alpha = \int d^2\theta\, d^2\bar{\theta}\; D^\alpha V \cdot W_\alpha.$$

For non-abelian gauge theories:

$$V = V_a T^a.$$

$$e^V \longmapsto e^{i\bar{\Lambda}} e^V e^{-i\Lambda}$$

WZ gauge:

$$e^V = 1 + V + \frac{1}{2}V^2$$

Gaugino superfield:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^{V}\right).$$

$$\bar{W}_{\dot\alpha} = -\frac{1}{4}DD\left(e^{V}\bar{D}_{\dot\alpha}e^{-V}\right).$$

Can show that

$$W_\alpha \longmapsto e^{i\Lambda}W_\alpha e^{-i\Lambda}.$$

In component fields, one can obtain:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V + \frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V]$$

$$\Rightarrow W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu D_\mu\bar{\lambda}(y))_\alpha$$

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu]$$

$$D_\mu = \partial_\mu - \frac{i}{2}[v_\mu, \cdot]$$

To introduce $g$ explicitly, redefine

$$V \to 2gV \quad \left(\begin{matrix} v_\mu \mapsto 2g v_\mu \\ \lambda \qquad \lambda \\ D \qquad D \end{matrix}\right)$$

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu]$$

$$D_\mu = \partial_\mu - ig[v_\mu, \cdot]$$

SYM Lagrangian:

$$\mathcal{L}_{SYM} = \frac{1}{32\pi}\,\mathrm{im}\left(\tau\int d^2\theta\ \mathrm{Tr}\,W^\alpha W_\alpha\right)$$

$$= \mathrm{Tr}\left(-\frac{1}{4}F^2 - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}D^2\right) + \frac{\theta_{YM}}{32\pi^2}g^2\,\mathrm{Tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu}$$

with

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}.$$

## Gauge-Matter Actions

still chiral.

$$\Phi \mapsto e^{i\Lambda}\Phi.$$

chiral

The KE gauge-inv. combination is

$$\bar{\Phi} e^V \Phi.$$

So we get the following.

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\, \bar{\Phi} e^V \Phi + \left(\int d^2\theta\, W(\Phi) + c.c.\right)$$

D-term:

$$\bar{\Phi} e^V \Phi = \bar{\Phi}\Phi + \bar{\Phi} V \Phi + \frac{1}{2}\bar{\Phi} V^2 \Phi.$$

$$(D_\mu\phi)^2 - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + \bar{F}F$$
$$+ \frac{i}{\sqrt{2}}\bar{\phi}\lambda\psi - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}\phi + \frac{1}{2}\bar{\phi}D\phi.$$

## Fayet-Iliopoulos

We can do D-term transforms like a total derivative

$$V^A \mapsto V^A - i\Lambda + i\bar{\Lambda}$$
$$D^A \mapsto D^A + \partial_\mu\partial^\mu(\cdots)$$

U(1) gauge trafo

$$\mathcal{L}_{FI} = \sum \xi_A \int d^2\theta d^2\bar{\theta}\, V^A = \frac{1}{2}\sum_A \xi_A D^A$$

Full Lagrangian:

$$\bar{F}_i = \frac{\partial \omega}{\partial \phi^i} \rightarrow D^a = -g\bar{\phi}T^a\phi - g\xi^a.$$

$$\mathcal{L} = \mathcal{L}_{SYM} + \mathcal{L}_{matter} + \mathcal{L}_{FI}.$$

$$i\sqrt{2}g\bar{\phi}\lambda\psi$$

$$-\frac{1}{2}\frac{\partial^2 \omega}{\partial\phi^i\partial\phi^j}\psi^i\psi^j$$

$$-V$$

with potential:

$$V(\phi,\bar{\phi}) = \frac{\partial \omega}{\partial \phi^i}\frac{\partial \omega}{\partial \bar{\phi}_i} + \frac{g^2}{2}\sum|\bar{\phi}_i(T^a)^i{}_j\phi^j + \xi^a|^2$$

$$= \bar{F}F + \frac{1}{2}D^2\Big| \geq 0$$

# Vacua

- vacuum: lowest-energy state / minimal energy state.
- Lorentz-invariant config.
  - ↳ so all derivatives & non-scalars $= 0$.
  - ↳ so the only thing left is the scalar potential
- 1:1 to minima of scalar potential.

## SUSY:

- $\langle \Omega | P^0 | \Omega \rangle \geq 0$.

  SUSY vacua $\xleftrightarrow{1:1}$ zeros of $V$.

- This means solving

$$\bar{F}_i(\phi) = 0 \qquad \text{F-flat } (W)$$

$$D^a(\phi, \bar{\phi}) = 0 \qquad \text{D-flat. (must exist)}$$

mod out by gauge-equiv. transformation.

- No radiative corrections to $W$.

  → need non-perturbative corrections.

# Supersymmetric Vacua

SUSY vacua $\longleftrightarrow$ zeros of $V$.

$$\begin{cases} \bar{F}_i(\phi) = 0. & \text{F-term.} \\ D^a(\phi, \bar{\phi}) = 0, & \text{D-term} \end{cases}$$

Procedure:

1) D-term flat directions.

$$D^a(\phi, \bar{\phi}) = 0.$$

2) $W$ exists $\Rightarrow$ F-term lifts

$$\bar{F}_i(\phi) = 0.$$

classical vacuum.

- flat directions — moduli
- generically non-SUSY theories have space of classical flat directions lifted by radiative corrections.

Classical Moduli Space:

$$\mathcal{M}_{cl} = \left\{ \frac{\langle \phi_i \rangle}{D^a = 0} \; \forall a \right\} / \text{gauge transf.}$$

$$\updownarrow$$

$$\mathcal{M}_{cl} = \frac{\langle \text{Gauge-inv. Operators } X_r(\Phi) \rangle}{\text{cl. relations}}.$$

## SQED.

$$\mathcal{L}_{SQED} = \frac{1}{32\pi} \operatorname{im}\left(\tau \int d^2\theta\, W_\alpha W^\alpha\right) + \int d^2\theta\, d^2\bar\theta \left(Q_i^\dagger e^{2V} Q_i + \tilde{Q}_i^\dagger e^{-2V} \tilde{Q}_i\right)$$

D-Term:

$$D = Q_i^\dagger Q^i - \tilde{Q}_i^\dagger Q^i = 0.$$

Gauge-inv.

$$Q^i \longmapsto e^{i\alpha} Q^i$$
$$\tilde{Q}^i \longmapsto e^{-i\alpha} \tilde{Q}^i$$

$\therefore$

$$\dim_{\mathbb{C}} \mathcal{M}_{cl} = 2F - 1.$$

Invariant:

$$M^i{}_j = Q^i \tilde{Q}_j$$

Meson matrix only has 1 nonv. e'value:

$$\varepsilon_{i_1 \dots i_F}\, \varepsilon^{j_1 \dots j_F}\, M^{i_1}{}_{j_1} \cdots M^{i_F}{}_{j_F} = 0$$

$$\det(M - \lambda \mathbb{1}) = \underbrace{\lambda^{F-1}(\lambda - \lambda_0)(-1)^F}_{\text{only } \lambda^F \,\&\, \lambda^{F-1} \text{ nonvanish}}$$

so

$$\epsilon_{i_1 \dots i_F} M^{i_1} M^{i_2} \epsilon^{j_1 \dots j_F} = 0.$$

$(F-1)^2$ ex. conditions.

$$\dim_{\mathbb{C}} \mathcal{M}_{cl} = 2F - 1.$$

Kähler metric

$$ds^2 = \frac{1}{2} \frac{1}{\sqrt{M^\dagger M}}\, dM\, dM^\dagger$$

( singular — dof light ~~appear there~~.

## Massless SQCD

| | $SU(N)$ | $SU(F)_L$ | $SU(F)_R$ |
|---|---|---|---|
| $Q^i{}_a$ | $N$ | $F$ | $1$ |
| $\tilde{Q}^b{}_j$ | $\bar{N}$ | $1$ | $F$ |

$$D^A = Q^{\dagger b}_{\ \ i}(T^A)^c_{\ b}\, Q^i_{\ c} - \tilde{Q}^b_{\ i}(T^A)^c_{\ b}\, \tilde{Q}^{\dagger i}_{\ \ c} = 0.$$

$$M^i_{\ j} = Q^i_{\ a}\tilde{Q}^a_{\ j}$$

with Kähler potential

$$K = 2\,\mathrm{Tr}\sqrt{M^\dagger M}$$

For $F<N$, can put

broken to $SU(N-F)$

$$Q = \left(\begin{array}{cccc|c} v_1 & & & & \\ & \ddots & & & \\ & & & v_F & \end{array}\right)$$

$M$ can be diagonalised in terms of $F$ ev. e'values $V_i$.

$\Rightarrow (2N-F)F$ massive gauge bosons.

$$\dim_{\mathbb{C}} \mathcal{M}_{cl} = F^2$$

$$= 2FN - \left[(N^2-1) - ((N-F)^2-1)\right]$$

take away gauge dof.

For $F \geq N$,

$$Q = \left(\begin{array}{ccc} v_1 & & \\ & \ddots & \\ & & v_F \\ \hline & & \end{array}\right)$$

$$\dim_{\mathbb{C}} \mathcal{M}_{cl} = 2NF - (N^2-1).$$

⊕

$$B = \varepsilon^{a_1 \dots a_N} Q^1_{\ a_1} \cdots Q^N_{\ a_N}$$

w/

$$\det M - B\tilde{B} = 0.$$

# Super-Higgs Mechanism

SQCD, $SU(N)$, $F$ flavours. all but $v_1 \neq 0$.

$$\# \text{ Broken generators} = N^2 - 1 - ((N-1)^2 - 1) = 2(N-1) + 1.$$

$$\text{Adj}_N = 1 \oplus \square \oplus \bar{\square} \oplus \text{Adj}_{N-1}$$

$$Q = \begin{pmatrix} \omega^0 & \psi \\ \omega & Q' \end{pmatrix}$$

$\psi$: $\square_{F-1}$, $\omega$: $\square_{N-1}$, $Q'$: $(\square, \square)$.

$$\tilde{Q} = \begin{pmatrix} \tilde{\omega}^0 & \tilde{\psi} \\ \tilde{\omega} & \tilde{Q}' \end{pmatrix}$$

b. dof

$$v_1 = 0 \Rightarrow \underbrace{2(N^2 - 1)}_{\text{gauge bosons}} + \underbrace{4NF}_{\text{cx. scalars}}.$$

$$v_1 \neq 0 \Rightarrow \underbrace{2((N-1)^2 - 1)}_{g.b.} + 4 \times \underbrace{1 + 2(N-1)}_{\text{massive dof}}$$

$\chi_1^\alpha, \chi_2^\alpha$ eaten $\omega, \tilde{\omega}$ $\oplus$ $\omega^0 - \tilde{\omega}^0$, $\chi^0$

$$+ \underbrace{4(N-1)(F-1)}_{\text{massless } Q'\tilde{Q}'} + \underbrace{2}_{\omega^0 + \tilde{\omega}^0} + \underbrace{4(F-1)}_{\psi, \tilde{\psi}}$$

## Gauge Kinetic Function

Really we have

$$\tau \int d^2\theta \, \mathrm{Tr} W^\alpha W_\alpha \longmapsto \int d^2\theta \, F_{ab}(\Phi) W^{\alpha a} W^b_\alpha$$

Adj ⊗ Adj. G-invariant.

$G \subset \mathrm{Iso}(\mathcal{M})$. $\mathcal{M}$ is scalar mfd.

$$K(\Phi^i, \bar{\Phi}_i) \longmapsto K(\Phi^i, (\bar{\Phi} e^{2gV})_i).$$

Geometrical operations on $\mathcal{M}$ in $N=1$ → unchanged by gauging isometries since $v$ + $s$ not in same multiplet.

Note: If dof does not propagate,

$$V\big|_{min} \longrightarrow W_i = 0$$

∴ SUSY-breaking has no dependence on Kähler potential.

## $\mathcal{N}=2$ SUSY

[vector] : $V=(\lambda_\alpha, A_\mu, D) \oplus \Phi = (\phi, \psi_\alpha, F)$.

[hyper] : $H_1 = (H_1, \psi_{1\alpha}, F_1) \oplus \bar{H}_2 = (\bar{H}_2, \bar{\psi}_{2\dot{\alpha}}, \bar{F}_2)$

- have R-symmetry $SU(2)_R$.
- no $W$ since this breaks $SU(2)_R$. potential D-term:

$$F^a = 0$$

$$D^a = -g[\phi, \bar{\phi}]^a.$$

$$\Rightarrow V = \frac{1}{2} g^2 \mathrm{Tr}[\phi, \bar{\phi}]^2.$$

huge moduli space of vacua.

@ generic point in moduli space:

- diagonalise $\phi$ so gauge group $U(1)^r$.
- LE: $r$ massless vectors

  $\dim G - r$ massive vectors.

  depend on VEVs.

$\Rightarrow$ <u>Coulomb Phase</u>

Hypermultiplets-

$H_1, \bar{H}_2$ form $SU(2)_R$-doublet.

$$V(H_1, H_2) = \frac{1}{2} g^2 |\bar{H}_1 T^a_R H_1 - \bar{H}_2 T^a_R H_2|^2$$

w/

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta} \, (\bar{H}_1 e^{2gV_R} H_1 + \bar{H}_2 e^{-2gV_R} H_2)$$

$$+ \int d^2\theta \sqrt{2} g H_1 \Phi H_2 + h.c.$$

~

To build NLSM note

$$\mathrm{Im}\,\mathcal{F}_{ab} \times \lambda^a \sigma^\mu D_\mu \bar{\lambda}^b \longleftrightarrow K_a^b \times \psi^a \sigma^\mu D_\mu \bar{\psi}_b$$

∴ Can combine into holo. prepotential,

$$\mathcal{F}_{ab} = \frac{\partial^2 F}{\partial \Phi^a \partial \Phi^b}$$

$$K = -\frac{i}{32\pi} \bar{\Phi}_a \frac{\partial F(\Phi)}{\partial \Phi^a}$$

Special Kähler Manifold, w/

$$V(\Phi, \bar{\Phi}) = -\frac{1}{2\pi} (\mathrm{im}\,\mathcal{F}_{ab})^{-1} [\bar{\Phi}, \mathcal{F}_c(\Phi) T^c]^a [\bar{\Phi}, \mathcal{F}_d(\Phi) T^d]^b .$$

$$\Big\} \oplus \text{Hypers}$$

$$\mathcal{M} = \mathcal{M}^V \otimes \mathcal{M}^H$$

Hyper Kähler manifold,

$W=4 \Rightarrow \mathcal{M} = \mathbb{R}^{6n}$ trivial topology.

## R-symmetry

- Defined: outer automorphism of Super-Poincaré

Useful for

- generator (necessary) in SCFT.
- superspace. (rotations in $\theta$)
- chiral modes.
- non-renormalisation thm.
- SUSY-breaking spontaneous.

① Constrains the algebra.

② Organises the supermultiplets.

③ Controls central charge $Z^{IJ}$

↪ short multiplets classified by R-reps.

↪ geometry of BPS states.

④ SCFT. — fix scaling dimensions