SUSY Class 3

3 Main topics today

- The free Wess Zumino action
- The superfield formalism
  ↳ relationship w/ the component formalism.
- NLSMs.

Free Wess-Zumino Model

Can we write down the easiest 4d model? ← This is from my old notes.

Let us give a first intro. Here.

$$S = \int d^4x \, (\mathcal{L}_{scalar} + \mathcal{L}_{fermion})$$

$$-\partial^\mu \phi^* \partial_\mu \phi \qquad i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi.$$

Clearly: $\begin{cases} \delta\phi = \epsilon\psi \\ \delta\phi^* = \varepsilon^\dagger \psi^\dagger \end{cases}$ if we want SUSY.

$$\delta \mathcal{L}_{scalar} = -\varepsilon \partial^\mu \psi \partial_\mu \phi^* - \varepsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi$$

Want up to total derivative.

$$\delta\psi_\alpha = -i(\sigma^\mu \varepsilon^\dagger)_\alpha \partial_\mu \phi$$
$$\delta\psi^\dagger_{\dot\alpha} = i(\varepsilon\sigma^\mu)_{\dot\alpha} \partial_\mu \phi^*.$$

$$\Rightarrow \delta\mathcal{L}_{fermion} = -\varepsilon \bar\sigma^\mu \sigma^\nu \partial_\nu \psi \partial_\mu \phi^* + \psi^\dagger \bar\sigma^\nu \sigma^\mu \varepsilon^\dagger \partial_\mu \partial_\nu \phi.$$

$$= -\delta\mathcal{L}_{scalar} - \partial_\mu(\ldots).$$

Is this SUSY? SUSY alg needs to close.

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2})\phi = i(-\varepsilon_1 \sigma^\mu \varepsilon_2^\dagger + \varepsilon_2 \sigma^\mu \varepsilon_1^\dagger)\partial_\mu \phi.$$

$-i\partial_\mu \longleftrightarrow P_\mu$ (!)

$$(\delta_{\epsilon_2} \quad \ldots)\psi_\alpha = -i(\sigma^\mu\epsilon_1^\dagger)_\alpha\,\epsilon_2\partial_\mu\psi + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\,\epsilon_1\partial_\mu\psi.$$

$$= i\left(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger\right)\partial_\mu\psi_\alpha$$

$$+ \underbrace{i\epsilon_{1\alpha}\epsilon_2^\dagger\bar\sigma^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\epsilon_1^\dagger\bar\sigma^\mu\partial_\mu\psi}_{\text{vanish on-shell.}}.$$

$\bar\sigma^\mu\partial_\mu\psi = 0.$

For off-shell need.

$$\mathcal{L}_{aux} = F^* F$$
$$\delta F = -i\epsilon^\dagger\bar\sigma^\mu\partial_\mu\psi$$
$$\delta F^* = i\partial_\mu\psi^\dagger\bar\sigma^\mu\epsilon$$

$$\Rightarrow \delta\mathcal{L}_{aux} = -i\epsilon^\dagger\bar\sigma^\mu\partial_\mu\psi F^* + i\partial_\mu\psi^\dagger\bar\sigma^\mu\epsilon F$$

$$\Rightarrow \delta\psi_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi + \epsilon_\alpha F$$
$$\delta\psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^* + \epsilon^\dagger_{\dot\alpha}F^*$$

But now

$$\delta\mathcal{L} = 0 \quad + \quad \delta_{[\epsilon_2}\delta_{\epsilon_1]}X = i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu X.$$

<u>DOF</u>

non-propagating.

	ϕ	ψ	F
ON –	2	2	0
OFF –	2	4	2

Can we get Noether:

$$\varepsilon J^\mu + \varepsilon^\dagger J^{\dagger\mu} = \sum_\alpha \delta X \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} - K^\mu$$

$\delta \mathcal{L} = \partial_\mu K^\mu$.

up to $\underset{\underset{0}{\shortparallel}}{\partial_\mu k^\mu}$ vector

$$\Rightarrow \quad J^\mu_\alpha = (\sigma^\nu \bar{\sigma}^\mu \psi)_\alpha \partial_\nu \phi^* . \qquad \partial_\mu J^\mu_\alpha = 0 .$$

$$J^{\dagger\mu}_{\dot\alpha} = (\psi^\dagger \bar{\sigma}^\mu \sigma^\nu)_{\dot\alpha} \partial_\nu \phi .$$

$$\hookrightarrow Q_\alpha = \sqrt{2} \int d^3\vec{x}\, J^0_\alpha$$

Then

$$[\varepsilon Q + \varepsilon^\dagger Q^\dagger , X] = -i\sqrt{2}\, \delta X .$$

Impose canonical com. rel.

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$

$$\{\psi_\alpha(\vec{x}), \psi^\dagger_{\dot\alpha}(\vec{y})\} = (\sigma^0)_{\alpha\dot\alpha}\, \delta^{(3)}(\vec{x}-\vec{y})$$

$$[\varepsilon_2 Q + \varepsilon_2^\dagger Q^\dagger , [\varepsilon_1 Q + \varepsilon_1^\dagger Q, X]] \; - \; \text{rew.}$$

$$= 2(\varepsilon_1 \sigma^\mu \varepsilon_2^\dagger - \varepsilon_2 \sigma^\mu \varepsilon_1^\dagger)\, i\partial_\mu X .$$

Now define.

$$H = \int d^3\vec{x}\, [\pi^* \pi + (\nabla\phi^*)\cdot(\nabla\phi) + i\psi^\dagger \bar{\sigma}\cdot\vec{\nabla}\psi]$$

$$\vec{P} = \int d^3\vec{x}\, (\pi\nabla\phi + \pi^*\nabla\phi^* + i\psi^\dagger \sigma \vec{\nabla}\psi]$$

$\therefore$ Generates spacetime translations

$$[P^\mu, X] = i\partial^\mu X.$$

$$\{$$

$$[\varepsilon_2 Q + \varepsilon_2^\dagger Q^\dagger, \varepsilon_1 Q + \varepsilon_1^\dagger Q^\dagger] = 2(\varepsilon_1 \sigma_\mu \varepsilon_2^\dagger - \varepsilon_2 \sigma_\mu \varepsilon_1^\dagger)\, P^\mu$$

$$\downarrow$$

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = -2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

$$\{Q_\alpha, Q_\beta\} = 0 = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\}$$

$$[Q_\alpha, P^\mu] = 0 = [Q^\dagger_{\dot\alpha}, P^\mu]$$

SUSY are global!

So we have realised the SUSY $N=1$ massless multiplet as a field theory. We note the following.

- The pseudoscalar field F is known as the auxiliary field. This is to fix the dof when the action is offshell.
  ↳ non-propagating field which closes the SUSY algebra w/out zero.

- this is not seen in the multiplet as these are just book-keeping devices.

- one can do that for gauge fields as well — this gives the vector multiplet

Superspace & Superfields

In the lectures the superspace is constructed as a homogeneous space (a coset space)
In particular, we had

$$g = \exp\left(i\left(\tfrac{1}{2}\omega^{\mu\nu}M_{\mu\nu} + x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot\alpha}\bar{Q}^{\dot\alpha}\right)\right) \in ISO(1,3|1)$$

Recall that we had constructed the coset space in the lectures as follows.

$$g = \exp(it^A T_A) \quad , \quad \text{split } g = h \oplus k$$
$$= \exp(i\omega^a M_a + i x^I K_I)$$

The space of left cosets requires $[h,k] \subset k$. So take representative

$$g_c(z) = \exp(i z^I K_I) \in G.$$

Then

$$g \cdot g_c(z) = g_c(z')\, h(g,z).$$

In Minkowski,

$$ISO(1,3) \cong SO(1,3) \ltimes \mathbb{R}^{1,3}. \Rightarrow \mathbb{R}^{1,3} \cong ISO(1,3)/SO(1,3)$$

The construction of superspace is then st. forward and follows very similarly from the Lorentz construction.

There is another way to understand what superspace is. Effectively,

→ superspace is a mfd w/ extra fermionic coords.

$$x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}.$$

→ abstractly, a superspace is a supermanifold constructed as a ringed space.
There are a bit more details in my notes, but I've digressed.

Superfields.

A superfield is a function that depends on superspace coordinates.

$$\Phi : M^{m|n} \longrightarrow X.$$

Let us contrast this with component fields.

$$f : M^{1,n-1} \longrightarrow X,\ x^*\Pi TX.$$

where the parity-reversed space gives.

$$\psi : M^4 \longrightarrow x^*\Pi TX.$$

This is a bit abstract so let us look at how it actually work.
Morally one can expand a superfield in Grassmann coordinates as follows.
We work in $\mathcal{N}=1$ 4d supersymmetry, so there are $\theta_1, \theta_2, \bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}$. so

$$Y(x,\theta,\bar{\theta}) = f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta\, m(x) + \bar{\theta}\bar{\theta}\, n(x) + \theta\sigma^\mu\bar{\theta}\ v_\mu(x)$$
$$+\theta\theta\ \bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\ \theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}\, d(x).$$

Using Berezin integration rules, we can now integrate over superspace.

Berezin Integration.

Note that for a Grassman variable,

$$\int d\theta = 0 \qquad \int d\theta\,\theta = 1.$$

$\theta = \delta_0(\theta)$, $d^2\theta = \frac{1}{2} d\theta^1 d\theta^2$, and.

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta\,.,\quad \int d^2\bar{\theta} = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}}$$

Q: How does superfield transform in superspace?

Recall:

$$\phi(x+a) = e^{-iaP}\phi(x)e^{iaP}. = \phi(x) - ia^\mu \underbrace{[P_\mu, \phi(x)]}_{i\partial_\mu\phi(x).}.$$

So in superspace SUSY-variations are generated by supercharges,

$$Y(x+\delta x,\dots) = e^{-i(\epsilon Q+\bar\epsilon\bar Q)}\, Y(x,\theta,\bar\theta)\, e^{i(\epsilon Q+\bar\epsilon\bar Q)}$$

can insert

$$e^{-i(xP+\theta Q+\bar\theta\bar Q)}\, Y(0,0,0)\, e^{i(\ )}$$

and use BCH to obtain

$$\begin{cases} \delta x^\mu = i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta \\ \delta\theta^\alpha = \epsilon^\alpha \\ \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha} \end{cases}$$

comes from $\{Q_\alpha,\bar Q_{\dot\alpha}\}\sim P_\mu$.

We can expand

$$\delta_{\epsilon,\bar\epsilon} Y = Y + i(\theta\sigma^\mu\bar\epsilon - \epsilon\sigma^\mu\bar\theta)\partial_\mu Y + \epsilon^\alpha\partial_\alpha Y + \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} Y$$

$$= Y - i\epsilon^\alpha[Q_\alpha, Y] + i\bar\epsilon^{\dot\alpha}[\bar Q_{\dot\alpha}, Y] + \dots$$

Then

$$\begin{cases} Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar Q_{\dot\alpha} = i\bar\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \end{cases}$$

<u>Relation with the component formalism</u>

SUSY Actions

I want actions. They are cool and give me theories.

How do I construct SUSY-invariant actions?

1). Use component fields, check SUSY-variations + add new component fields.

   $\Rightarrow$ This was the WZ action

2). We use superfields.

⚠ We only use superfields in $N=1$ rigid SUSY. (Normally only in 4d).
There are a few reasons for that, see discussion later.

Let's discuss how to build SUSY actions.

- $\int d\theta\,\theta$ translationally inv.

$$\int d(\theta+\xi)(\theta+\xi) = 1.$$

So.

$$\delta_{\epsilon\bar\epsilon}\int d^4x\, d^2\theta\, d^2\bar\theta\; Y = \int d^4x\, d^2\theta\, d^2\bar\theta\; \underbrace{\delta_{\epsilon,\bar\epsilon} Y}.$$

$$\epsilon^\alpha\partial_\alpha Y + \bar\epsilon_{\dot\alpha}\bar\partial^{\dot\alpha} Y + \partial_\mu\left[-i(\epsilon\sigma\bar\theta-\theta\sigma\bar\epsilon)Y\right]$$

no $\theta,\bar\theta$

total derivative.

So

$$\int d^4x\, d^2\theta\, d^2\bar\theta\; Y \qquad \text{is automatically SUSY-invariant.}$$

- Integrating Grassman variables give

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \mathcal{A} = \int d^4x\, \mathcal{L}(x).$$

in general some product of superfields.

- Basic superfield Y has too many field components to be an irrep. of SUSY.

  → need extra constraints.

- There are 2 such constraints.

  1) Chiral ⇒ Chiral.

  Introduce covariant derivatives

$$\begin{cases} D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\beta}\bar{\theta}^{\dot\beta}\partial_\mu . \\ \bar{D}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} + i\theta^\beta \sigma^\mu_{\beta\dot\alpha}\partial_\mu . \end{cases} \Rightarrow \{D_\alpha, \bar{D}_{\dot\beta}\} = -2\sigma^\mu_{\alpha\dot\beta}P_\mu .$$

  see comment.

  These are precisely the right action on chiral superspace.
  and hence

$$\delta_{\epsilon,\bar\epsilon}(D_\alpha Y) = D_\alpha(\delta_{\epsilon,\bar\epsilon} Y).$$ ⇒ SUSY-inv. constraint.

  c.f. $[\bar{\partial}_{\dot\alpha}, \epsilon Q] = \epsilon^\beta \sigma^\mu_{\beta\dot\alpha}\partial_\mu$ generates a $\theta\theta\bar\theta\bar\theta$ term.

  ⇒ not SUSY-inv. $\bar{\partial}_{\dot\alpha} Y$.

Chiral superfields are

$$\bar{D}_{\dot\alpha}\Phi = 0. \qquad D_\alpha \bar{\Phi} = 0$$

antichiral.

This is NOT real.

We can continue by moving into chiral superspace.

$$\begin{cases} y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}. \\ \bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}. \end{cases}$$

$$\therefore \quad \bar{D}_{\dot{\alpha}}\theta_\beta = \bar{D}_{\dot{\alpha}} y^\mu = 0.$$

$\left(i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right)$! $\Rightarrow$ left-inv. vector fields in superspace.

Then can expand

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y)$$

$$\Rightarrow \Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\,\partial_\mu\phi(x) - \theta\theta F(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta}$$
$$- \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\,\Box\phi(x).$$

You can check SUSY transforms.

$$\delta_{\epsilon,\bar{\epsilon}}\Phi = (i\epsilon Q + i\bar{\epsilon}\bar{Q})\Phi(y,\theta).$$

$$Q_\alpha = -i\partial_\alpha$$
$$\bar{Q}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu}$$

You can apply this to $\delta\Phi$,

$$\delta_{\epsilon,\bar{\epsilon}}\Phi = \left(\epsilon^\alpha\partial_\alpha + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\frac{\partial}{\partial y^\mu}\right)\Phi(y,\theta)$$

$$= \underbrace{\sqrt{2}\epsilon\psi}_{\delta\phi} + \sqrt{2}\theta\underbrace{\left(-\sqrt{2}\epsilon F + \sqrt{2}i\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\phi\right)}_{\delta\psi_\alpha} - \theta\theta\underbrace{\left(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\frac{\partial}{\partial y^\mu}\psi\right)}_{\delta F.}$$

This is Q3.

2). Real constraint $\Rightarrow$ Vector Superfield.

We define

$$V = \bar{V}$$

which gives the general expansion:

$8_B + 8_F$.
$\Rightarrow 4_B + 4_F$ @ gauge-fix
$\Rightarrow 2_B + 2_F$ @ on-shell.

$$V(x,\theta,\bar\theta) = C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \theta\sigma^\mu\bar\theta\, v_\mu + \frac{i}{2}\theta\theta\,(M(x) + iN(x))$$
$$- \frac{i}{2}\bar\theta\bar\theta\,(M(x) - iN(x)) + i\theta\theta\bar\theta\left(\bar\lambda(x) + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right)$$
$$- i\bar\theta\bar\theta\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(D(x) - \frac{1}{2}\partial^2 C(x)\right)$$

In particular we have the following transformation,

$$V \longmapsto V + \Phi + \bar\Phi$$
$$v_\mu \longmapsto v_\mu - \partial_\mu(2\,\mathrm{Im}\,\phi)$$

Gauge transformation! The cpts transform

$$\begin{cases} C \mapsto C + 2\,\mathrm{Re}\,\phi \\ \chi \mapsto \chi - i\sqrt{2}\,\psi \\ M \mapsto M - 2\,\mathrm{Im}\,F \\ N \mapsto N + 2\,\mathrm{Re}\,F \\ v^\mu \mapsto v^\mu - 2\partial_\mu \mathrm{Im}\,\phi \\ D, \lambda \text{ inv} \end{cases} \Rightarrow \quad \begin{array}{c} \underline{\text{gauge fix}} \\ \mathrm{Re}\,\phi = -\frac{C}{2} \\ \psi = -\frac{i}{\sqrt{2}}\chi \\ \mathrm{Re}\,F = -\frac{N}{2} \\ \mathrm{Im}\,F = \frac{M}{2} \end{array}$$

Wess-Zumino gauge.

$$\Rightarrow V_{WZ} = \theta\sigma^\mu\bar\theta\, v_\mu(x) + i\theta\theta\,\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\,\theta\lambda(x) + \underbrace{\frac{1}{2}\theta\theta\bar\theta\bar\theta D(x)}_{\text{auxiliary field}}.$$

WZ gauge is defined @

$$C = M = N = \chi = 0. \quad \text{no restrictions on } v^\mu.$$

The gauge transf

$$\phi = -\bar\phi\ , \ \psi = 0\ , \ F = 0$$

still remains.

Note:

- $V_{WZ}^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\, v_\mu v^\mu$, $V_{WZ}^n = 0 \quad n \geq 3$.
- WZ gauge not SUSY. Need compensating transf. with Φ after SUSY transformation.

3) Supercurrents.

We can incorporate composite operators. ⇒ conserved currents + SUSY currents.

Linear Superfield

Recall

$$Q = \int d^3x\, j^0$$

⇝ real scalar superfield satisfying

before it's just $\theta\bar{\theta}$ dependence

$$D^2 J = \bar{D}^2 J = 0$$

constrains spacetime dependence of some fields by imposing EOM in x-space.

⇒ on-shell constraint.

This gives

$$J = J(x) + i\theta j(x) - i\bar{\theta}\bar{j}(x) + \theta\sigma^\mu\bar{\theta}\, j_\mu(x) + \frac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu j(x) - \frac{1}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{j}(x)$$
$$+ \frac{1}{4}\theta^2\bar{\theta}^2 J(x).$$

Then

$$\partial^\mu j_\mu = 0$$

- less independent components $\because D^2 J = 0$
- no spins ≥ 1 , otherwise cannot gauge j^μ w/out new gauge fields
  - ↪ so J spin-0.
- J not unique & up to Schwinger terms. Since

$$[Q_\alpha, Q] = 0.$$

$$\Rightarrow [Q_\alpha, j_\mu] = Q_{\alpha\mu} \,.\ \therefore\ \partial_\mu Q_\alpha^\mu = 0$$

But

$$\int d^4x\, [Q_\alpha, j_0] = 0 \qquad O_{\alpha 0} = \partial^\nu A_{\alpha\nu}.$$

$$\parallel \qquad\qquad \Uparrow$$

$$\int dt\, [Q_\alpha, Q]$$

This is a Schwinger term. → gives diff. completions of $\mathcal{J}$.
Above we chose

$$O_{\alpha\mu} = -2i(\sigma_{\mu\nu})_\alpha{}^\beta\, \partial^\nu j_\beta.$$

Supersymmetry current

Associated to fermionic charge

$$Q_\alpha = \int d^3x\, S_\alpha^0$$

$$\Rightarrow \{\bar{Q}_{\dot\alpha}, S_{\alpha\nu}\} = 2\sigma^\mu_{\alpha\dot\alpha} T_{\mu\nu} + O_{\alpha\dot\alpha\nu}$$

since $\nu=0$ do not integrate to 0, but gives $\{Q, \bar{Q}\} \sim P$.
$\Rightarrow$ get extra $T_{\mu\nu}$.

Schwinger term gives diff. completions, eg. Ferrara-Zumino multiplet

$$2\bar{D}^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha} J_\mu = D_\alpha X.$$

If $X=0 \Rightarrow$ SCFT.

eg. R-multiplet.

$$2\bar{D}^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha} \underbrace{R_\mu}_{\text{real}} = \underbrace{\chi_\alpha}_{\text{chiral}}$$

Matter Actions

Now let us construct actions. Recall we have shown that $\delta S = 0$ for SUSY Lagrangians constructed from superfields. We can however have

$$\delta \mathcal{L} = \partial_\mu V^\mu_{SUSY}.$$

Let's look at what terms to put. Use SUSY-transf. on general multiplet

① D-terms

$$\delta D = \partial_\mu \left(-\epsilon \sigma^\mu \bar{\lambda} + \bar{\epsilon} \bar{\sigma}^\mu \lambda \right)$$

$$\Rightarrow \quad \mathcal{L}_D = \frac{1}{2} D^{S^L}$$

So construct real superfield S^L and extract D-term,

$$\mathcal{L}_D = \int d^2\theta \, d^2\bar{\theta} \, S^L.$$

② F-terms

Note

$$\delta F = \partial_\mu \left(i\sqrt{2} \, \bar{\epsilon} \bar{\sigma}^\mu \psi \right)$$

So this is known as the F-term Lagrangian

$$S_F = \int d^4x \int d^2\theta \, F^{\Phi}$$

If we only consider chiral superfields, we will have.

$$[\theta] = M^{-\frac{1}{2}}$$

so $\theta^2 \bar{\theta}^2$ component has dim. $[Y] + 2$.

$$[\Phi] = M^1 \quad \Rightarrow \quad [\Phi \bar{\Phi}|_D] = M^4 \checkmark$$

Can write unconstrained:

$$\int d^2\theta\, d^2\bar\theta\, \bar\Phi\Phi = \int d^2\bar\theta\, \bar\Phi \underbrace{D^2\Phi}.$$

$D^2\Phi = 0$ gives EOM.

The most general form

$$\int d^2\theta\, d^2\bar\theta\, K(\Phi, \bar\Phi) \qquad [K] = 2.$$

- K not fn of $D_\alpha\Phi$ as this gives higher derivatives > 2.
- most general expression

$$K(\Phi,\bar\Phi) = \sum c_{mn}\Phi^m\bar\Phi^n \qquad c_{mn} = c^*_{nm}$$

$$c_{mn} \sim \Lambda^{2-(m+n)} \quad \text{renorm.}$$

- Kähler potential is Kähler-transform invariant.

$$K \longmapsto K + \Lambda(\Phi) + \bar\Lambda(\bar\Phi).$$

For interactions, we will need

$$\int d^4x\, d^2\theta\, W(\Phi) \quad \text{— chiral superfield.}$$

$$\wr$$

$$W(x,\theta,\bar\theta) = e^{i\theta\sigma^\mu\bar\theta\partial_\mu}\Sigma(x,\theta)$$

In fact,

$$\int d^4x\, d^2\theta\, d^2\bar\theta\, Y = \frac{1}{4}\int d^4x\, d^2\theta\, \underbrace{\bar D^2 Y}_{\text{manifestly chiral}}$$

but going backwards is not true $\because$ D may not exist.

holo fn: $\bar{D}_{\dot{\alpha}} W = \frac{\partial W}{\partial \Phi} \bar{D}_{\dot{\alpha}} \Phi = 0$

$$\Rightarrow \quad \mathcal{L}_{int} = \int d^2\theta \, W(\Phi) + \int d^2\bar{\theta} \, \bar{W}(\bar{\Phi}).$$

$$W = \sum a_n \Phi^n \qquad [W] = 3.$$

$$\theta^2 = [W] + 1.$$

W is constrained by R-symmetry.

$$W(\Phi) = W(\phi) + \sqrt{2} \frac{\partial W}{\partial \phi} \theta \psi + \theta\theta \left(\frac{\partial W}{\partial \phi} F + \frac{1}{2} \frac{\partial^2 W}{\partial \phi \partial \phi} \psi\psi \right)$$

$[\Phi] = 1 \Rightarrow [W] = 3$

$\hookrightarrow$ at most cubic. for renormalisability

So in general we write

$$K(\Phi^i, \bar{\Phi}^i) = \bar{\Phi}_i \Phi^i$$

$$W(\Phi^i) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k.$$

Potential on shell is $\left(\bar{F} = \frac{\partial W}{\partial \phi} \right)$

$$V(\phi, \bar{\phi}) = \left| \frac{\partial W}{\partial \phi} \right|^2 = \bar{F}F. \qquad \text{F-term potential.}$$

NLSMs:

Allow for more general terms (non-renormalizable terms) in the theory exposes connection bt. geometry + QFTs.

Let us start with NLSMs.

Def An n-dim σ-model is a QFT encoded by geometric data.

Describes config. spaces that are mapping spaces into geometric space

$$\phi \in \mathrm{Map}(\underbrace{\Sigma}_{\text{config.}}, \underbrace{X}_{\text{target}})$$

The Lagrangian is then

$$\mathcal{L} = -\eta^{\mu\nu} g_{ab} \partial_\mu \varphi^a \cdot \partial_\nu \varphi^b$$

eg. Relativistic particles

$$\phi : \Sigma \longrightarrow M^{1,3}$$

Gauge field $S^1 \to X$, $A \in \Omega^1(X)$.

Config. space

$$C^\infty(\mathbb{R}, X) / \mathrm{Diff}(\mathbb{R}).$$

Linear σ-models. $\phi : \Sigma \to M$

$M^{1,3}$ $\mathbb{R}^n$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)\cdot(\partial^\mu \phi) + \frac{1}{2}\mu^2 \phi^2 - \frac{\lambda}{4}(\phi\cdot\phi)^2$$

$$\phi_i = R_{ij}\phi_j \quad O(N) \text{ transform.}$$

$\Rightarrow$ X is a Riemannian mfd in general.

What does SUSY bring to the table?

Prop. The target space of 4d $\mathcal{N}=1$ SUSY field theory is a Kähler manifold.

A Kähler manifold is a specific manifold. Let us illustrate.

- It is a cx. mfd w/ an integrable cx. structure J.
- It Hermitian mfd equipped w/ metric g.

$$ds^2 = g_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}}$$

There is a natural $(1,1)$-form.

$$J = i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^{\bar{\nu}}$$

A symplectic mfd is compatible w/ cx. structure I

$$\omega(IX, IY) = \omega(X,Y)$$

$$\Rightarrow g(X,Y) = \omega(X, IY).$$

Kähler is all 3. $\rightarrow$ natural 2-form J closed.

also $\exists$ scalar K on each patch s.t.

$$J = i\partial\bar{\partial}K.$$

So how does SUSY F.T. give a Kähler mfd? Recall

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K + \left(\int d^2\theta\, W + \text{c.c.}\right)$$

We can think about going all the way & calculating all the terms. In particular write

$$W(\Phi) = W(\phi) + W_i \Delta^i + \tfrac{1}{2} W_{ij} \Delta^i \Delta^j$$

$$\Delta^i = \Phi^i - \phi^i = \sqrt{2}\theta\psi^i(y) - \theta\theta F(y)$$

Then

F: $\quad -\omega_i F^i - \frac{1}{2}\omega_{ij}\psi^i\psi^j$

D:

Define $\Delta^i(x) = \Phi^i - \varphi^i(x)$, $\bar{\Delta}_i = \bar{\Phi}_i - \bar{\varphi}_i(x)$

$$K = K(\varphi,\bar{\varphi}) + K_i\Delta^i + \text{c.c.} + \frac{1}{2}K_{ij}\Delta^i\Delta^j + \frac{1}{2}\bar{K}^{ij}\bar{\Delta}_i\bar{\Delta}_j + K_i{}^j\Delta^i\bar{\Delta}_j$$
$$+ \dots$$

Up to total derivatives we can extract

$$\int d^2\theta d^2\bar{\theta}\, K = K_i{}^{\bar{j}}\left(F^i\bar{F}_j + \partial_\mu\varphi^i\partial^\mu\bar{\varphi}_j - \frac{i}{2}\psi^i\sigma^\mu\partial_\mu\bar{\psi}_j + \frac{i}{2}\partial_\mu\psi^i\sigma^\mu\bar{\psi}_j\right)$$
$$+ \frac{i}{4}K^k_{ij}\left(\psi^i\sigma^\mu\bar{\psi}_k\partial_\mu\varphi^j + \psi^j\sigma^\mu\bar{\psi}_k\partial_\mu\varphi^i - 2i\psi^i\psi^j\bar{F}_k\right)$$
$$-\frac{i}{4}K_k^{ij}(\text{c.c.})$$
$$+\frac{1}{4}K^{kl}_{ij}\psi^i\psi^j\bar{\psi}_k\bar{\psi}_l$$

only $K_{i\bar{j}}$ enter
so invariant under
Kähler transf.

Important $\to$ all the terms in Lagrangian are geometric. Use

$$F^i = (K^{-1})^i{}_{\bar{k}}\bar{\omega}^k - \frac{1}{2}(K^{-1})^i{}_{\bar{k}}K^k{}_{lm}\psi^l\psi^m.$$

$$\therefore\ \mathcal{L} = K_{i\bar{j}}\left(\partial_\mu\varphi^i\partial^\mu\bar{\varphi}^{\bar{j}} + \frac{i}{2}D_\mu\psi^i\sigma^\mu\bar{\psi}^{\bar{j}} - \frac{i}{2}\psi^i\sigma^\mu D_\mu\bar{\psi}^{\bar{j}}\right) - (K^{-1})_{i\bar{j}}\,\omega^i\bar{\omega}^{\bar{j}}$$
$$-\frac{1}{2}\left(\omega_{ij} - \Gamma^{\bar{k}}_{ij}\omega_{\bar{k}}\right)\psi^i\psi^j - \text{c.c.}$$
$$+\frac{1}{4}R^{\bar{k}\bar{l}}{}_{ij}\,\psi^i\psi^j\bar{\psi}^{\bar{k}}\bar{\psi}^{\bar{l}}$$

All specified by geometry.

$$D_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{jk}\partial_\mu\varphi^j\psi^k.$$

$\mathcal{N}=2 \to$ Special-Kähler

$\mathcal{N}=4 \Rightarrow \mathcal{M} = \mathbb{R}^{6n}.$